Seip's differentiability concepts as a particular case of the Bertram – Glöckner – Neeb construction

by

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Abstract. From the point of view of unification of differentiation theory, it is of interest to note that the general construction principle of Bertram, Glöckner and Neeb leading to a C^k differentiability concept from a given C^0 one, besides subsuming the Keller – Bastiani C_c^k differentiabilities on real Hausdorff locally convex spaces, also does the same to the "arc-generated" interpretation of the Lipschitz theory of differentiation by Frölicher and Kriegl, and likewise to the "compactly generated" theory of Seip's continuous differentiabilities. In this article, we give the details of the proof for the assertion concerning Seip's theory. We also give an example indicating that the premises in Seip's various inverse and implicit function theorems may be too strong in order for these theorems to have much practical value. Also included is a presentation of the BGN – setting reformulated so as to be consistent with the Kelley – Morse – Gödel – Bernays – von Neumann type approach to set theory, as well as a treatment of the function space constructions and development of their basic properties needed in the proof of the main result.

Introduction and some preliminaries

According to the introductions in [16] and [17], Seip's main motivation for the development of his theory of differentiation was to establish a setting with a "purely" topological basis where function spaces also beyond Banach spaces could be considered as domains and ranges of differentiable maps, and where also some kind of "cartesian closedness" holds so that "exponential laws" would be available to facilitate proving smoothness of given maps. Seip grounded his theory on the observation by Gabriel and Zisman [7; p. 47] that considering the k-extension of the compact open topology for the set of continuous functions between topological spaces gives rise to a cartesian closed category.

A bit less imprecisely, the last assertion means the following. For a topological space X let kX denote the space with the same underlying set equipped with the finest topology such that the identity restricted on every compact set is continuous $X \to kX$. Call X *compactly generated* iff it is Hausdorff with $kX = X$. Let $X \sqcap Y = kP$ when P is the usual product space of X and Y, and let $Y^X = kC$ when C is the compact open topological space of all continuous functions $X \to Y$. Then $u \mapsto \hat{u}$, where $\hat{u} : (x, y) \mapsto u(y)(x)$, defines a bijection $\left(Z^X\right)^Y \to Z^{X \sqcap Y}$ for any compactly generated X, Y, Z.

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In order to be still less imprecise, note that actually we do not have a bijection as just written, but from the *underlying set* of $\big(Z^{X}\big)^{Y}$ onto that of $Z^{X \sqcap Y}$, that is $\sigma_{\mathrm{rd}}\big(\big(Z^{X}\big)^{Y}\big) \to \sigma_{\mathrm{rd}}\big(Z^{X \sqcap Y}\big)$. However, it follows that the same function $u \mapsto \hat{u}$ is a homeomorphism $\big(Z^{X}\big)^{Y} \to Z^{X \sqcap Y}$.

In Example 56 below, the use of this kind of exponential laws $\big(Z^{X}\big)^{Y} \approx Z^{X \sqcap Y}$ in Seip's theory of differentiation is exemplified by giving a simple proof of Seip – smoothness $E \sqcap E \to kE$ of the map $(x, y) \mapsto x \circ (\iota + y)$, where we have the locally convex test function space $E = \mathcal{D}(I\!R)$ with $\iota = \mathrm{id}\, I\!R$.

It is remarkable that the method of Example 56 applies although the locally convex space E is neither a canonical function space in Seip's theory, nor compactly generated by [6; Theorem 6.1.4 (iii) , Proposition 6.2.8 (ii) , pp. 190, 195] . Note that by a *canonical function space* in a theory of differentiation we mean any prearranged object of the theory which has as its underlying set the set of all order i differentiable functions $f : E \supseteq U \to F$ for some fixed E, F, U, i. Having intrinsic exponential laws in a theory requires canonical function spaces. If there are no such, possible exponential laws have to established in an ad hoc manner, as for example in (new) particular cases of the general theory developed in [2] .

However, it should be noted that although exponential laws may provide easy proofs of smoothness of maps between spaces of smooth functions, the same does not hold for maps between spaces of finite order differentiable functions. Since practical inverse and implicit function theorems may require some kind of use of Banach spaces, and since spaces of smooth functions seldom are such, we see that the goals of on one hand possessing exponential laws, and on the other hand having available usable inverse or implicit function theorems may be somewhat contradictory.

Indeed, for example in [16; pp. 55 – 57, 73, 80, 81, 92, 93] , inverse and implicit function theorems were provided. However, they were so formulated that their proofs within the theory became almost trivial, this having the consequence that verifying their presuppositions in practice became almost impossible. For instance, if with $I = [\,0\,, 1\,]$ we consider the diffeomorphism $f : C^{\infty}(I) = G \to G$ defined by $x \mapsto \varphi \circ x$ for a fixed nonaffine smooth diffeomorphism $\varphi : I\!R \to I\!R$, we shall see in Example 57 below that *f is not scharf differenzierbar* at any constant point $x = I \times \{\xi\}$ when $\xi \in I\!R$ is such that $\varphi''(\xi) \neq 0$.

In the literature, there do not seem to be any serious applications of Seip's theory of differentiation. Besides the facts given above, one possible reason for Seip's theory not having become popular may be the overwhelmingly "categorical" style of presentation in [16] and [17] where also things more simply expressible without any notions of category theory have been stated in such terms. There also are some obscurities in the notations for the various categories.

A good example of making simple matters obscure and complicated by jargonizing them category-theoretically is [16; Definition 8.6, p. 108] where a *norm* in a real locally convex Hausdorff space E with topology $\mathcal{T}$ just means any $n \in \mathcal{C}^{A}$ satisfying $z \in n(z) \subseteq \mathrm{Cl}_{\mathcal{T}}(n(x) + n(z - x))$ for all $x, z \in A$ when A is the set of vectors of E and $\mathcal{C}$ is the set of bounded absolutely convex closed sets in E. A simple example of this kind of (rather useless ?) "norm" is $A \ni x \mapsto \{\, t\,x : -1 \leq t \leq 1\}$. See also Example 58 below for some more details on this matter.

Seip's theory of continuous differentiabilities in [16] and [17] is one in the line of the theories of differentiation where some canonical function spaces are incorporated in the theory wishing to obtain certain intrinsic exponential laws. In these

theories, there generally are no natural inverse and implicit function theorems. Other examples in this line can be found in [1], [5], [6] and [15].

In the other line, one has theories more adapted to handling inverse or implicit function theorems, but function spaces generally have to be treated in an ad hoc manner. One example is the Banach space calculus around the classical continuous differentiabilities sketched in [3; pp. 147 ff., 181 − 182]. Other examples are our modification in [10; pp. 237 − 241] of [5], generalizing [3], and the particular cases in [8] of [2] which also generalizes [16] and [6], and a portion of [1]. Note also [18; pp. 3 − 6, 22 ff.] and [9; pp. 73 ff., 140 − 143] where quite special differentiability concepts are designed hoping to get certain "intrinsic" inverse and implicit function theorems applicable e.g. to some maps of Fréchet function spaces.

In this article, we concentrate on proving that indeed Seip's theory can be obtained as a particular case of the general construction in [2]. We hope to give the proof of the corresponding assertion associated with the theory in [6] later.

To one acquainted with the developments in [1], [16] and [2], because of the manner the higher order differentiabilities are constructed in the last, it should not be a surprise that portions of the theories in the former are obtainable as particular cases of the construction in [2].

Namely, one gets the higher order differentiabilities in [2] by a recursion using the difference quotient family associating with a continuous $f : E \to F$ the map $f^{[1]}$: "$E \times E \times \mathbb{K}$" $\to F$ given by $(x, u, t) \mapsto t^{-1}(f(x + t\,u) - f(x))$ for t invertible. In [1] and [16], one instead uses the derivative $f' : E \to L$ given by $f'(x)\,u = f^{[1]}(x, u, 0)$ with L a space of continuous linear maps $E \to F$ having the exponential law property that a map $E \to L$ is continuous iff the associated map "$E \times E$" $\to F$ is such. Differentiability being defined with the aid of a remainder condition, cf. [1; Définition 2.1, p. 39] and [16; Definition 4.2, p. 60], requiring continuity of the map $(x, u, t) \mapsto f^{[1]}(x, u, t) - f'(x)\,u$, it is expectable that the different approaches lead to the same concepts.

Although the results are not surprising, it is surprising that the proofs, when properly presented, are not at all straightforward or trivial. This may be considered as a consequence of the different modes of development in the theories.

The contents of the present article consisting of a single section divided into this introduction and four subsections is roughly described by the following list of the titles of these subsections:

In A we reformulate the basic setting in [2] in order to get it accordant with the logically economical and precise notational system we followed in [11] and to be complemented below. In B we give the basic definitions and establish the basic facts concerning general compactly generated (vector) spaces and spaces of continuous functions needed in the sequel. In C we give a short account of the matter in the title to make things precise. In D we establish the main result of this article, given as the conjunction of Theorems 52 and 55 below.

At the end of D we have included two examples of which Example 56 shows how the use of exponential laws in Seip's theory can be utilized to get simple proofs of smoothness of certain kind of maps. The purpose Example 57 is to raise the

question whether any serious applications of Seip's various inverse and implicit function theorems are possible.

In the above introduction, we used some notations more or less informally, which we abandon from now on. For example, we above let "$u(x)$" denote the function value of u at x. From now on, it is $u`x$. Precisely, for any classes $u, x$ we define $u`x = \bigcap\{ y : \forall z\, ;\, (x,z) \in u \Leftrightarrow y = z \}$ deviating from [13; Definition 68, p. 261] in order to have $R`x = \mathbf{U}$ also e.g. if $(x,y), (x,z) \in R$ with $y \neq z$. However, note that $u`x = u(x)_{\mathrm{Kelley}}$ for any *function* u and any class x.

Likewise, the symbol '$\sqcap$' no longer refers to any compactly generated product. Instead, we define $E \sqcap F = ((\sigma_{\mathrm{rd}} E) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}} F), (\tau_{\mathrm{rd}} E) \underset{\mathrm{t}}{\times} (\tau_{\mathrm{rd}} F))$ so that (see below) if E, F are topologized $\mathbf{K}$–modules, then $E \sqcap F$ is the product of the underlying algebraic modules equipped the usual Tihonov product of the respective topologies. In particular, this applies when E, F are real topological vector spaces.

Besides the preceding two ones, from now on we generally use also the other notational conventions of [11]. In addition to these, we give the following

1 *Conventions.* Working in a set theoretical setting closely (but not exactly) paralleling the presentation in [13; Appendix, pp. 250–281], see also [14], and defining $\mathbf{U} = \{ x : x = x \}$ and $\{x\} = \{ z : \forall y\, ;\, x \in y \Rightarrow x = z \}$ and $\{x, y\} = \{x\} \cup \{y\}$, we have [$\forall z\, ;\, z \in \{x\} \Leftrightarrow x = z$] if $x \in y$ for some y, i.e. if x is a set, otherwise having $\{x\} = \mathbf{U}$, i.e. for any set z it holds that $z \in \{x\}$.

Deviating from [13; Definition 48, p. 259], we define $(x, y) = \{\{x, y\}, \{y\}\}$, and call P an *ordered pair* if and only if there are sets x, y with $P = (x, y)$. If x or y is not a set, it follows that $(x, y) = \mathbf{U}$.

Adapting [13; Definitions 51, 52, p. 259], we define

$$\sigma_{\mathrm{rd}}\, z = \bigcup\bigcup z \setminus \bigcup\bigcap z \cup \bigcap\bigcup z \quad \text{and} \quad \tau_{\mathrm{rd}}\, z = \bigcap\bigcap z\,.$$

Then $x = \sigma_{\mathrm{rd}} P$ and $y = \tau_{\mathrm{rd}} P$ hold for any ordered pair $P = (x, y)$. Furthermore, we have $\sigma_{\mathrm{rd}} \mathbf{U} = \tau_{\mathrm{rd}} \mathbf{U} = \mathbf{U}$, since for example

$$\sigma_{\mathrm{rd}} \mathbf{U} = \bigcup\bigcup \mathbf{U} \setminus \bigcup\bigcap \mathbf{U} \cup \bigcap\bigcup \mathbf{U} = \bigcup \mathbf{U} \setminus \bigcup \emptyset \cup \bigcap \mathbf{U} = \mathbf{U} \setminus \emptyset \cup \emptyset = \mathbf{U}\,.$$

We define $(x, y, z) = ((x, y), z)$ and $(x\,; y, z) = (x, (y, z))$ and $(x, y\,; u, v) = (x, y, (u, v))$ and $\sigma^2_{\mathrm{rd}} Z = \sigma_{\mathrm{rd}}(\sigma_{\mathrm{rd}} Z)$ and $\tau\sigma_{\mathrm{rd}} Z = \tau_{\mathrm{rd}}(\sigma_{\mathrm{rd}} Z)$. In an obvious manner, one may continue to get a succession of definitions for instance for (x, y, z, u) and $(x\,; y, z, u)$ and $\sigma^3_{\mathrm{rd}} Z$ and $\tau\sigma^2_{\mathrm{rd}} Z$ and $\tau^2\sigma_{\mathrm{rd}} Z$, etc.

With the aid of the preceding conventions, we can give a precise meaning to the concept of map. Any set $\mathcal{T}$ with $\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{T} \} \cup \{ U \cap V : U, V \in \mathcal{T} \} \subseteq \mathcal{T}$ being called a *topology*, a *topological space* is any ordered pair X such that $\tau_{\mathrm{rd}} X$ is a topology with $\sigma_{\mathrm{rd}} X = \bigcup \tau_{\mathrm{rd}} X$. Letting $f[A] = f``A = \{ y : \exists x \in A\, ;\, (x, y) \in f \}$ and $f```\mathcal{A} = \{ f``A : A \in \mathcal{A} \}$ and $\mathcal{T} \cap A = \{ U \cap A : U \in \mathcal{T} \}$, by a *topological map* (or topological morphism) we mean any ordered pair $\tilde{f}$ such that topologies $\mathcal{T}$ and $\mathcal{U}$ and a function f exist with $f \subseteq (\bigcup \mathcal{T}) \times (\bigcup \mathcal{U})$ and $f^{-\iota}```\mathcal{U} \subseteq \mathcal{T} \cap (\mathrm{dom}\, f)$ and $\tilde{f} = (\mathcal{T}, \mathcal{U}, f)$. A topological map $\tilde{f}$ is called *global* iff $\bigcup \sigma^2_{\mathrm{rd}} \tilde{f} \subseteq \mathrm{dom}\, \tau_{\mathrm{rd}} \tilde{f}$.

We say that f is *continuous* $\mathcal{T} \to \mathcal{U}$ if and only if $\tilde{f} = (\mathcal{T}, \mathcal{U}, f)$ is a topological map. Equivalently, we may also say that $\tilde{f}$ is continuous, or a continuous map, or that $f : \mathcal{T} \to \mathcal{U}$ is continuous, or a continuous map.

Note that we above did not want to mix the concept of topological space in that of topological map because this only would have made things more complicated. We only introduced the concept of topological space because in some (quite rare) connections we may be able to simplify wordings by using such a concept.

Instead of topological maps, we shall below mainly consider module or vector maps defined as follows. A *ring* (structure) on $I\!K$ is any ordered pair $\mathbf{K}$ such that functions $\mathfrak{a}, \mathfrak{c} : I\!K^{\times 2\cdot} = I\!K \times I\!K \to I\!K$ exist with $\mathbf{K} = (\mathfrak{a}, \mathfrak{c})$ and satisfying the usual ring postulates when we let $s + t = \mathfrak{a}`(s, t)$ and $s\,t = \mathfrak{c}`(s, t)$. The ring $\mathbf{K}$ is called *commutative* iff $\mathfrak{c}$ is such, meaning that $\{\,(r, s, t) : (s, r, t) \in \mathfrak{c}\,\} \subseteq \mathfrak{c}$ holds, and *unital* if and only if it possesses a unity 1, meaning that 1 is a $\mathfrak{c}$ – identity, i.e. that $\emptyset \neq \{\,(r, s, t) : \{\,r, s\,\} = \{\,t, 1\,\}\,\} \mid I\!K^{\times 2\cdot} \subseteq \mathfrak{c}$, and we also have $1 \neq o$ when we let $o = \mathbf{0}_{\mathbf{K}} = \bigcap\{\,s : (s, s, s) \in \sigma_{\mathrm{rd}}\mathbf{K}\,\}$. Let $\boldsymbol{\upsilon}_s X = \mathrm{rng}\,\sigma_{\mathrm{rd}} X$.

A *structured* $\mathbf{K}$ *– module* on S is any ordered pair E such that $\mathbf{K}$ is a commutative unital ring and functions $a : S^{\times 2\cdot} \to S$ and $c : (\boldsymbol{\upsilon}_s\mathbf{K}) \times S \to S$ exist with $\sigma_{\mathrm{rd}} E = (a, c)$ and satisfying the usual module postulates when we let $x + y = a`(x, y)$ and $t\,x = c`(t, x)$. Defining $\upsilon_s E = \mathrm{rng}\,\sigma_{\mathrm{rd}}^2 E$, now a $\mathbf{K}$ *– vector map* is any ordered pair $\tilde{f}$ such that structured $\mathbf{K}$ – modules E, F and a function f exist satisfying $f \subseteq (\upsilon_s E) \times (\upsilon_s F)$ and $\tilde{f} = (E, F, f)$. A vector map $\tilde{f}$ we agree to say to be *global* if and only if $\upsilon_s \sigma_{\mathrm{rd}}^2 \tilde{f} \subseteq \mathrm{dom}\,\tau_{\mathrm{rd}} \tilde{f}$ holds.

If in some connection we wish to be perfectly explicit about the structured (vector space or) module with respect to whose linear structure the algebraic operations in a given linear combination are to be taken, we shall use a notational device giving for example $(x + t\,u)_{\mathrm{svs}\,E} = \sigma_{\mathrm{rd}}^2 E`(x, \tau\sigma_{\mathrm{rd}} E`(t, u))$ instead of the ambiguous "$x + t\,u$". In a similar fashion, we have for example $[\,\{t\}\,A + B + C\,]_{\mathrm{svs}\,E} = \sigma_{\mathrm{rd}}^2 E\,[\,\sigma_{\mathrm{rd}}^2 E\,[\,\tau\sigma_{\mathrm{rd}} E\,[\,\{t\} \times A\,] \times B\,] \times C\,]$. If in such a connection we also have a function f with $\mathrm{dom}\,f \subseteq \upsilon_s E$, in view of [13; Theorem 69, p. 261] then writing $y = f`((x + t\,u)_{\mathrm{svs}\,E}) \neq \mathbf{U}$ is equivalent to writing $x, u \in \upsilon_s E$ and $t \in I\!K$ and $x + t\,u \in \mathrm{dom}\,f$ and $y = f`(x + t\,u)$ in the looser convention.

In the preliminaries to [11] we gave the formal definition of the two topology Tihonov product $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U}$. We omit the definition of the corresponding product $\prod_{\mathrm{ti}} \boldsymbol{\mathcal{T}}$ of an arbitrary small family $\boldsymbol{\mathcal{T}}$ of topologies. The vector space product of two algebraic modules X, Y is $X \underset{\mathrm{vs}}{\times} Y$. The corresponding product of a small family $\boldsymbol{X}$ is $\prod_{\mathrm{VS}\,I\!K} \boldsymbol{X}$ when each $X \in \mathrm{rng}\,\boldsymbol{X}$ is a $\mathbf{K}$ – module with $I\!K = \boldsymbol{\upsilon}_s\mathbf{K}$.

We further let $X^{\,\Omega]_{\mathrm{vs}}} = \prod_{\mathrm{VS}\,\mathrm{dom}^2 \tau_{\mathrm{rd}} X} (\Omega \times \{X\})$. If X is a module (structure) and S is a submodule (set) therein, then $X_{|\,S} = (\sigma_{\mathrm{rd}} X \mid S^{\times 2\cdot}, \tau_{\mathrm{rd}} X \mid (\mathbf{U} \times S))$ is the corresponding submodule structure. We let $E_{/S} = (\sigma_{\mathrm{rd}} E_{\;|\,S}, \tau_{\mathrm{rd}} E \cap S)$.

For convenience, we modify our notation "$\langle\, t_z : z \in S\,\rangle$" from [11] as follows.

We let $\langle\, \mathfrak{T} : \mathfrak{y}_1, \ldots\, \mathfrak{y}_l : \mathfrak{x}\,\mathfrak{E}\,\rangle = \{\,\mathfrak{z} : \exists\, \mathfrak{x}, \mathfrak{x}_1, \ldots\, \mathfrak{x}_k\,;\, \mathfrak{z} = (\mathfrak{x}, \mathfrak{T})$ and $\mathfrak{x}\,\mathfrak{E}\,\}$ when $\mathfrak{x}, \mathfrak{x}_1, \ldots\, \mathfrak{x}_k, \mathfrak{y}_1, \ldots\, \mathfrak{y}_l, \mathfrak{z}$ are distinct variable symbols and $\mathfrak{T}$ is a term and $\mathfrak{E}$ is an expression such that the expression $\mathfrak{x}\,\mathfrak{E}$ is a formula having the variable symbol $\mathfrak{x}$ in the first place, and the common free variables in $\mathfrak{T}$ and $\mathfrak{x}\,\mathfrak{E}$ distinct from $\mathfrak{x}, \mathfrak{y}_1, \ldots\, \mathfrak{y}_l$ are exactly $\mathfrak{x}_1, \ldots\, \mathfrak{x}_k$, and $\mathfrak{z}$ does not occur free in $\mathfrak{T}$ or $\mathfrak{x}\,\mathfrak{E}$. In the case where $\mathfrak{y}_1, \ldots\, \mathfrak{y}_l$ is an empty list, we let $\langle\, \mathfrak{T} : \mathfrak{x}\,\mathfrak{E}\,\rangle = \langle\, \mathfrak{T} :: \mathfrak{x}\,\mathfrak{E}\,\rangle$. Thus for example the class

$$f = \langle\, \mathrm{e}^{\,s+t} : x = (s, t) \in I\!R \times I\!R\,\rangle$$
$$= \{\, z : \exists\, x, s, t\,;\, z = (x, \mathrm{e}^{\,s+t}) \text{ and } x = (s, t) \in I\!R \times I\!R\,\}$$

is the function defined on $\mathrm{dom}\,f = I\!R \times I\!R$, and whose value at $x = (s, t) \in I\!R \times I\!R$ is $f`x = \mathrm{e}^{\,s+t}$, and which one might express by $f : I\!R \times I\!R \ni (s, t) \mapsto \mathrm{e}^{\,s+t}$. For $g = \langle\, \mathrm{e}^{\,t} : t : x = (s, t) \in I\!R \times I\!R\,\rangle = \{\, z : \exists\, x\,;\, z = (x, \mathrm{e}^{\,t})$ and $x = (s, t) \in I\!R \times I\!R\,\}$, we have $g = \{\,(s, t, \mathrm{e}^{\,t})\}$ if $s, t \in I\!R$, and $g = \emptyset$ otherwise.

When compared to our old convention, for example, if f is a function, then $f = \langle\, f`x : x \in \mathbf{U}\,\rangle_{\mathrm{old}} = \langle\, f`x : x \in \mathrm{dom}\,f\,\rangle_{\mathrm{old}} = \langle\, f`x : f : x \in \mathrm{dom}\,f\,\rangle$, whereas $\langle\, f`x : x \in \mathrm{dom}\,f\,\rangle = \{\, z : \exists\, x, f\,;\, z = (x, f`x)$ and $x \in \mathrm{dom}\,f\,\} = \mathbf{U}^{\times 2\cdot}$.

Recall that a *function* (or family) is any $f \subseteq \mathbf{U}^{\times 2.}$ such that no x, y, z exist with $(x, y), (x, z) \in f$ and $y \neq z$. We generally put $R^{-\iota} = \{ (y, x) : (x, y) \in R \}$ and $g \circ f = \{ (x, z) : \exists\, y \,;\, (x, y) \in f \text{ and } (y, z) \in g \}$. We shall also utilize the definitions

$$f^{\vee} = \{ (x, u) : \emptyset \neq u = \{ (y, z) : (x, y, z) \in f \} \},$$
$${}^{\vee}f = \{ (y, u) : \emptyset \neq u = \{ (x, z) : (x, y, z) \in f \} \},$$
$$f_1{}^{\wedge} = \{ (x, y, z) : \exists\, u \,;\, (x, u) \in f_1 \text{ and } (y, z) \in u \},$$
$${}^{\wedge}f_1 = \{ (x, y, z) : \exists\, u \,;\, (y, u) \in f_1 \text{ and } (x, z) \in u \},$$
$$\mathrm{pr}_1 = \{ (x, y, x) : x, y \in \mathbf{U} \}, \qquad \mathrm{pr}_2 = \{ (x, y, y) : x, y \in \mathbf{U} \}.$$

The large *evaluation family* is $\mathrm{ev} = \{ (x, u, y) : u \text{ function and } (x, y) \in u \}$. The evaluation family at x is $\mathrm{ev}_x = \{ (u, y) : u \text{ function and } (x, y) \in u \}$.

Note that $0. = \emptyset$ and $1. = \{\emptyset\}$ and $2. = \{ \emptyset, 1. \}$ and $I\!N = I\!N_{\mathrm{o}} \setminus 1.$, and that $k + 1. = k^{+} = k \cup \{k\}$ for $k \in I\!N_{\mathrm{o}}$, and also that $\infty = I\!N_{\mathrm{o}}$ and $\infty^{+} = I\!N_{\mathrm{o}} \cup \{\infty\}$. If $\boldsymbol{x} \in \mathbf{U}^{k}$ and $\boldsymbol{y} \in \mathbf{U}^{l}$ and $k, l \in I\!N_{\mathrm{o}}$, then $\boldsymbol{x}\hat{\ }\boldsymbol{y} = \boldsymbol{x} \cup (\boldsymbol{y} \circ \langle\, k + j : j \in l \,\rangle^{-\iota})$.

For simplicity, we allow the notational inconsistency that for extended real numbers $s \leq t$ we have $[\, s, t \,] = \{ r : s \leq r \leq t \}$, whereas for functions f, g we define $[\, f, g \,] = \{ (x \,; y, z) : (x, y) \in f \text{ and } (x, z) \in g \}$. Proper notations for these would be for example $[\, s, t \,]_{\mathrm{i}}$ and $[\, f, g \,]_{\mathrm{f}}$, respectively.

We generally (but not exclusively, for example $x + y \cdot z \neq (x + y) \cdot z$ usually) apply the rule for reduction of parentheses given by the schema $\mathfrak{T}_k\, \mathrm{b}_k \,\ldots\, \mathfrak{T}_1\, \mathrm{b}_1\, \mathfrak{T}_0 = (\mathfrak{T}_k\, \mathrm{b}_k \,\ldots\, \mathfrak{T}_1)\, \mathrm{b}_1\, \mathfrak{T}_0$ when the $\mathfrak{T}_\iota$ are terms and b_ι are binary symbols. Also, we usually understand monadic symbols to act prior to the binary ones. For example, above we have $\sigma_{\mathrm{rd}}\, z = ((\bigcup (\bigcup z)) \setminus (\bigcup (\bigcap z))) \cup (\bigcap (\bigcup z))$. Further, usually we let $\mathrm{m}_k \ldots \mathrm{m}_0\, \mathfrak{T}\, \mathrm{n}_0 \ldots \mathrm{n}_l = ((\mathrm{m}_k \ldots (\mathrm{m}_0\, \mathfrak{T}))\, \mathrm{n}_0) \ldots \mathrm{n}_l$ when the m_ι and n_ι are monadic. For example $\bigcup \Phi^{-\iota} = (\bigcup \Phi)^{-\iota} = \{ (y, x) : \exists\, f \,;\, (x, y) \in f \in \Phi \}$, and usually $\bigcup \Phi^{-\iota} \neq \bigcup (\Phi^{-\iota})$.

2 *Remark.* If we wished to give systematically our rules for the reduction of parentheses so that the use of such vague words as "usually" could be avoided, we should give an exhaustive list of our function symbols, the symbols there being grouped according to their intended "level", like in [14; Theory of notation, 0.30 − 0.48, pp. 15 − 21], cf. the "type" and "power" there. The preceding rules then would be applied without exception when symbols on the same level are considered, whereas symbols on a higher level would act prior to those on a lower one.

For example $\mathrm{lev}\,`\cdot\text{'} > \mathrm{lev}\,`+\text{'}$ and $\mathrm{lev}\,`{}^{\mathrm{fct\,exp}}\text{'} > \mathrm{lev}\,`\text{``}\text{'}$ so that we have $r + s \cdot t = r + (s \cdot t)$ and $f\,\text{``}\,B^{k} = f\,\text{``}(B^{k}) = f\,[\,B^{k}\,]$, here $`{}^{\mathrm{fct\,exp}}\text{'}$ being the invisible binary function symbol the result of whose application to 'B' 'k' we write "B^{k}".

Since we accept this kind of "nonlinear" expressions which are discarded in [14], our theory of notation, if properly presented, is much more complicated than the one in [14; pp. 15 − 26]. Note further that although $\mathrm{lev}\,`{}^{\mathrm{fct\,exp}}\text{'} > \mathrm{lev}\,`+\text{'}$, we have $B^{k+l} = B^{(k+l)}$, and generally $B^{k+l} \neq B^{k} + l$ unless $k = l = \emptyset$.

3 *Remark.* Suppose, cf. [14; p. 59], that one introduces the ordered *Weihe pairs* $(x, y)_{\mathrm{we}} = x \,\boldsymbol{,}\, y = (\mathcal{P}_s x) \times (\mathcal{P}_s y) = \{ (u, v) : u \subseteq x \text{ and } v \subseteq y \}$ having the property that the implication $[\; x \,\boldsymbol{,}\, y = u \,\boldsymbol{,}\, v \Rightarrow x = u \text{ and } y = v \;]$ *holds for any classes* x, y, u, v, and not only for such sets as in the case of our (reversed) Wiener pairs, and further the ordered Weihe tuples by the recursion schema $(x_k, \ldots x_1, x_0)_{\mathrm{we}} = x_k \,\boldsymbol{,}\, \ldots x_1 \,\boldsymbol{,}\, x_0 = (x_k \,\boldsymbol{,}\, \ldots x_1) \,\boldsymbol{,}\, x_0$. Then one could manage with very few binary symbols and with no higher order ones except the Weihe tuple symbols themselves.

Indeed, for example given a ${}_k$ – place function symbol $\mathfrak{g}$ and introducing the monadic function symbol $\mathfrak{f}$ by the definition

$$\mathfrak{f}\,x = \{\, u : \forall\, x_1, \ldots\, x_k \,;\, x = x_1 \,,\ \ldots\, x_k \Rightarrow u \in \mathfrak{g}\, x_1\ \ldots\, x_k \,\}\,,$$

one could totally eliminate $\mathfrak{g}$ from one's set theoretic edifice.

A. A reformulation of the general BGN – setting

In [2; Definition 1.1, pp. 219 – 220], postulates are given for a (generally large) family $[\,\mathcal{C}^{\,0}, \boldsymbol{\tau}\,] : (E\,,F\,) \mapsto (\mathcal{C}^{\,0}\text{\`{}}(E\,,F\,)\,, \boldsymbol{\tau}\text{\`{}}(E\,,F\,))$ where further $\mathcal{C}^{\,0}\text{\`{}}(E\,,F\,)$ is a (small) family $\tau_{\rm rd}\,E \ni U \mapsto \mathcal{C}^{\,0}\text{\`{}}(E\,,F\,)\text{\`{}}U$, the last one being a set of functions $U \to \upsilon_s F$ which are continuous $\tau_{\rm rd}\,E \to \tau_{\rm rd}\,F$, and where $\boldsymbol{\tau}\text{\`{}}(E\,,F\,)$ is a topology for the set $(\upsilon_s E) \times (\upsilon_s F)$. We reformulate these postulates in the next

4 ***Definitions.*** By a *topologized* $\mathbf{K}$ – *module* we mean any structured $\mathbf{K}$ – module E such that $(\upsilon_s E\,, \tau_{\rm rd}\,E\,)$ is a topological space. If $\mathbf{K}$ is a division ring, or a field, we speak of *topologized vector spaces*. Let now $\mathbf{K}$ be a commutative ring with unity 1, and let $\mathcal{O}$ be a class of topologized $\mathbf{K}$ – modules with $\boldsymbol{K} = (\mathbf{K}\,, \tau_{I\!\!K}) \in \mathcal{O}$. Letting $F^{\,/E} = \{\, f : f$ a function and $f \subseteq (\upsilon_s E) \times (\upsilon_s F\,)\,\}$, we consider a class $\mathcal{C}_0$ having as its members some maps $\tilde{f} = (E\,, F\,, f\,) \in \mathcal{O}^{\times 2\cdot} \times \mathbf{U}$ where $f \in F^{\,/E}$. We require each $\tilde{f} \in \mathcal{C}_0$ to be continuous with open domain, which is expressed shortly by the inclusion $\tau_{\rm rd}\,\tilde{f}^{\,-\iota\,\text{```}}\,\tau^{\,2}\sigma_{\rm rd}\,\tilde{f} \subseteq \tau\sigma_{\rm rd}^{\,2}\,\tilde{f}$. Letting

$$\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod} = \{\,(E\,,F,G) : E\,,F \in \mathrm{dom}^{\,2}\,\mathcal{C}_0 \text{ and } \mathcal{C}_0\text{-}\mathrm{prod}_{\,\rm mcl}\,(E\,,F,G)\,\}\,,$$

where $\mathcal{C}_0\text{-}\mathrm{prod}_{\,\rm mcl}\,(E\,,F,G)$ denotes the condition that

$$\sigma_{\rm rd}\,G = (\sigma_{\rm rd}\,E\,) \underset{\rm vs}{\times} (\sigma_{\rm rd}\,F\,) \text{ and } (G, E\,, \mathrm{pr}_1\,|\,\upsilon_s G), (G, F, \mathrm{pr}_2\,|\,\upsilon_s G) \in \mathcal{C}_0$$
$$\text{and } \forall\, f\,, g\,, H\,;\, (H, E\,, f\,), (H, F, g) \in \mathcal{C}_0 \Rightarrow (H, G, [\,f\,, g\,]\,) \in \mathcal{C}_0\,,$$

we now say that $\mathcal{C}_0$ is a *productive class* on $\mathcal{O}$ over $\boldsymbol{K}$ iff in addition to the above assumptions, we also have $\mathcal{O}^{\times 2\cdot} \subseteq \mathrm{dom}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod})$. A productive class $\mathcal{C}_0$ on $\mathcal{O}$ over $\boldsymbol{K}$ we say to be a BGN – *class* iff with the inversion $\iota = \{\,(s\,,t\,) : (s\,,t\,,1) \in \tau_{\rm rd}\,\mathbf{K}\,\}$, for all $E\,, F, G \in \mathcal{O}$ and for all $f\,, g\,, y\,, P, U$, we also have the following

(1) $P = (E\,, F\,)$ and f is a function and
$[\,\forall\, Z\,;\, \exists\, h\,;\, Z \in f \Rightarrow Z \in h \subseteq f$ and $(P, h) \in \mathcal{C}_0\,] \Rightarrow (P, f\,) \in \mathcal{C}_0\,,$

(2) $(E\,, F\,, f\,), (F\,, G, g) \in \mathcal{C}_0 \Rightarrow (E\,, G, g \circ f\,) \in \mathcal{C}_0\,,$

(3) $U \in \tau_{\rm rd}\,E$ and $y \in \upsilon_s F \Rightarrow (E\,, E\,, \mathrm{id}_{\rm v} E), (E\,, F\,, U \times \{y\}) \in \mathcal{C}_0\,,$

(4) $(\boldsymbol{K}, \boldsymbol{K}, \iota\,) \in \mathcal{C}_0\,,$

(5) $f\,, g \in \mathcal{C}_0\text{``}\{\,(\boldsymbol{K}, F\,)\,\}$ and $\mathrm{dom}\,f = \mathrm{dom}\,g$ and $f \circ \iota \circ \iota \subseteq g \Rightarrow f = g\,,$

(6) $(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}\text{\`{}}(E\,, E\,), E\,, \sigma_{\rm rd}^{\,2}\,E\,), (\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}\text{\`{}}(\boldsymbol{K}, E\,), E\,, \tau\sigma_{\rm rd}\,E\,) \in \mathcal{C}_0\,.$

5 Lemma. *If $\mathcal{C}_0$ is a productive class on $\mathcal{O}$ over $\boldsymbol{K}$, it then holds that* $\mathrm{dom}\,\mathcal{C}_0 \subseteq \mathrm{dom}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}) = \mathcal{O}^{\times 2\cdot}$, *and that $\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}$ is a function $\mathcal{O}^{\times 2\cdot} \to \mathcal{O}$.*

Proof. Let $\mathcal{C}_0$ be a productive class on $\mathcal{O}$ over $\boldsymbol{K}$. To see that $\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}$ is a function, letting $(E\,, F, G), (E\,, F, H) \in \mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}$, then $\sigma_{\rm rd}\,G = (\sigma_{\rm rd}\,E\,) \underset{\rm vs}{\times} (\sigma_{\rm rd}\,F\,) = \sigma_{\rm rd}\,H$. If we take $f = \mathrm{pr}_1\,|\,\upsilon_s G$ and $g = \mathrm{pr}_2\,|\,\upsilon_s G$, then $\mathrm{id}_{\rm v} G = [\,f\,, g\,] = \mathrm{id}_{\rm v} H$ with $(G, H, \mathrm{id}_{\rm v} G), (H, G, \mathrm{id}_{\rm v} G) \in \mathcal{C}_0$, whence by the assumption that every $\tilde{f} \in \mathcal{C}_0$ is continuous we get $\tau_{\rm rd}\,H \subseteq \tau_{\rm rd}\,G \subseteq \tau_{\rm rd}\,H$, hence $G = H$. To get $\mathrm{dom}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}) = \mathcal{O}^{\times 2\cdot}$, note that $\mathcal{O}^{\times 2\cdot} \subseteq \mathrm{dom}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}) \subseteq (\mathrm{dom}^{\,2}\,\mathcal{C}_0)^{\times 2\cdot} \subseteq \mathcal{O}^{\times 2\cdot}$. We trivially have $\mathrm{rng}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}) \subseteq \mathrm{dom}^{\,2}\,\mathcal{C}_0 \subseteq \mathcal{O}$ and $\mathrm{dom}\,\mathcal{C}_0 \subseteq \mathcal{O}^{\times 2\cdot}$. $\square$

6 Proposition. *Let $\mathcal{C}_0$ be a* BGN – *class on $\mathcal{O}$ over $\boldsymbol{K}$. Then* $\mathrm{dom}\,\mathcal{C}_0 = \mathcal{O}^{\times 2\cdot}$, *and for all $E\,, F, G \in \mathcal{O}$ and for all $f\,, g$ with $\mathit{\Pi} = \mathcal{C}_0\text{-}\mathrm{P}_{\rm rod}\text{\`{}}(F, G)$ it holds that*

(a) $(\mathit{\Pi}, F, \mathrm{pr}_1 \,|\, \upsilon_s \mathit{\Pi}), (\mathit{\Pi}, G, \mathrm{pr}_2 \,|\, \upsilon_s \mathit{\Pi}) \in \mathcal{C}_0$,
(b) $(E, F, f), (E, G, g) \in \mathcal{C}_0 \Rightarrow (E, \mathit{\Pi}, [\, f, g\,]) \in \mathcal{C}_0$.

Proof. Lemma 5 gives $\mathrm{dom}\, \mathcal{C}_0 \subseteq \mathcal{O}^{\times 2\cdot}$, and $\mathcal{O}^{\times 2\cdot} \subseteq \mathrm{dom}\, \mathcal{C}_0$ follows from (3) of Definitions 4 above. If $F, G \in \mathcal{O}$ and $\mathit{\Pi} = \mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}\grave{}\,(F, G)$, since by Definitions 4 we have $(F, G) \in \mathcal{O}^{\times 2\cdot} \subseteq \mathrm{dom}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}})$, and since by Lemma 5 we know that $\mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}$ is a function $\mathcal{O}^{\times 2\cdot} \to \mathcal{O}$, it follows that $(F, G, \mathit{\Pi}) \in \mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}$, and consequently that $\mathcal{C}_0\text{-}\mathrm{prod}_{\mathrm{mcl}}(F, G, \mathit{\Pi})$ holds. This directly gives the asserted (a) and (b) above. $\square$

Note that for the validity of (a) and (b) in Proposition 6 above it in fact suffices that $\mathcal{C}_0$ only is a productive class on $\mathcal{O}$ over $\boldsymbol{K}$. However, without the other BGN postulates, especially (2) of Definitions 4 above, these are of little use. For this reason, we gave preference to the preceding formulation.

Note also that by $\iota \circ \iota \subseteq \mathrm{id}$ we could have given (5) of Definitions 4 the following longer equivalent formulation: we have $f = g$ whenever $\{(\boldsymbol{K}, F)\} \times \{f, g\} \subseteq \mathcal{C}_0$ and $\mathrm{dom}\, f = \mathrm{dom}\, g$ and $f\grave{}\,t = g\grave{}\,t$ for all $\tau\sigma_{\mathrm{rd}}\, \boldsymbol{K}$–invertible t. This corresponds to the *determination axiom* [2; III, p. 220]. Our (4) corresponds to [2; I.4, p. 220], and (1) to [2; I.5, p. 220]. For a further comparison, see also Remark 11 below.

7 *Definitions.* For any classes $\mathcal{C}_0, \boldsymbol{K}, \tilde{f}$ generally letting

(1) $\mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}} = {\mathcal{C}_0}^{\times 2\cdot} \cap \{(\tilde{f}, \tilde{h}) : \forall E, F, f\,;\, \exists E_2, G, h\,;\, \forall h^0\,;\, \tilde{f} = (E, F, f) \Rightarrow \tilde{h} = (G, F, h)$ and $(E, E, E_2), (E_2, \boldsymbol{K}, G) \in \mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}$ and $[\, h^0 = \{(x, u, t, y) : f\grave{}\,((x + t\,u)_{\mathrm{svs}\,E}) = (f\grave{}\,x + t\,y)_{\mathrm{svs}\,F} \neq \mathbf{U}\} \Rightarrow \mathrm{dom}\, h^0 \subseteq \mathrm{dom}\, h$ and $h \subseteq h^0\,]\,\}$,

(2) $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} = \bigcap\{\tilde{h} : \forall \tilde{h}_1\,;\, \tilde{h} = \tilde{h}_1 \Leftrightarrow (\tilde{f}, \tilde{h}_1) \in \mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}\}$,

(3) $\mathcal{D}^k_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K}) = \{\tilde{f} : k \in \infty^+$ and $\exists\, \boldsymbol{f}\,;\, (\emptyset, \tilde{f}) \in \boldsymbol{f} \in {\mathcal{C}_0}^{k+1\cdot}$ and $\forall i \in k\,;\, (\boldsymbol{f}\grave{}\,i, \boldsymbol{f}\grave{}\,i^+) \in \mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}\}$,

we then say that $\tilde{f}$ is k^{th} order $\mathcal{C}_0-_{\mathrm{BGN}}$ *differentiable* over $\boldsymbol{K}$ if and only if we have $\tilde{f} \in \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K})$. We may say *first* instead of $1.^{\mathrm{th}}$, and *second* instead of $2.^{\mathrm{th}}$, etc. For $\tilde{f} \in \mathbf{U}$, we may call any $\tilde{f}^1 \in \mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}$“$\{\tilde{f}\}$ a first order BGN – *difference quotient* map for $\tilde{f}$ in $\mathcal{C}_0$ over $\boldsymbol{K}$. In case $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} \neq \mathbf{U}$, we may call $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f}$ *the* first order BGN – difference quotient map for $\tilde{f}$ in $\mathcal{C}_0$ over $\boldsymbol{K}$.

8 Lemma. *For any* $\mathcal{C}_0, \boldsymbol{K}, \tilde{f}$ *either* $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} \in \mathcal{C}_0$ *or* $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} = \mathbf{U}$. *Furthermore* $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} \in \mathcal{C}_0$ *holds iff a unique* $\tilde{f}^1$ *exists with* $(\tilde{f}, \tilde{f}^1) \in \mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}$.

Proof. Letting $\mathcal{H} = \{\tilde{h} : \forall \tilde{h}_1\,;\, \tilde{h} = \tilde{h}_1 \Leftrightarrow (\tilde{f}, \tilde{h}_1) \in \mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}\}$, one quickly verifies that either there is a set $\tilde{h}$ with $\mathcal{H} = \{\tilde{h}\}$ or $\mathcal{H} = \emptyset$. In the former case we have $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} = \bigcap \mathcal{H} = \tilde{h} \in \mathrm{rng}\,(\mathcal{C}_0\text{-}\mathrm{Difq}^1_{\boldsymbol{K}}) \subseteq \mathrm{rng}\,({\mathcal{C}_0}^{\times 2\cdot}) \subseteq \mathcal{C}_0$, and in the latter $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} = \bigcap \emptyset = \mathbf{U}$. The assertions now easily follow. $\square$

If $\tilde{f}$ is first order $\mathcal{C}_0-_{\mathrm{BGN}}$ differentiable over $\boldsymbol{K}$, then $\tilde{f}$ has at least one first order BGN – difference quotient map in $\mathcal{C}_0$ over $\boldsymbol{K}$. In the case where $\mathcal{C}_0$ is a BGN – class over $\boldsymbol{K}$, there is only one such, namely $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f}$, as follows from the next

9 Proposition. *If* $\mathcal{C}_0$ *is a* BGN – *class over* $\boldsymbol{K}$ *and* $\tilde{f}$ *is first order* $\mathcal{C}_0-_{\mathrm{BGN}}$ *differentiable over* $\boldsymbol{K}$, *then* $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f}$ *is the unique first order* BGN – *difference quotient map for* $\tilde{f}$ *in* $\mathcal{C}_0$ *over* $\boldsymbol{K}$.

Proof. Indeed, supposing that $\tilde{f}^{\iota}$ for ${}^{\iota = 1, 2}$ are first order difference quotient maps for $\tilde{f}$ in $\mathcal{C}_0$ over $\boldsymbol{K}$, we first get $\sigma_{\mathrm{rd}}\,\tilde{f}^1 = \sigma_{\mathrm{rd}}\,\tilde{f}^2$ since $\mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}$ is a function.

Putting $h_\iota = \tau_{\mathrm{rd}}\,\tilde{f}^{\iota}$, from $\mathrm{dom}\,h^0 \subseteq \mathrm{dom}\,h_\iota$ and $h_\iota \subseteq h^0$, we obtain $\mathrm{dom}\,h_1 = \mathrm{dom}\,h^0 = \mathrm{dom}\,h_2$, whence it suffices to prove that for any $W = (x,u,t) \in \mathrm{dom}\,h^0$ we have the equality $h_1`W = h_2`W$. Let $f = \tau_{\mathrm{rd}}\,\tilde{f}$.

To obtain $h_1`W = h_2`W$, putting $\gamma = \langle\,(x,u,s) : s \in \upsilon_s\boldsymbol{K}\,\rangle$, we consider $\gamma_\iota = h_\iota \circ \gamma$. Then $\mathrm{dom}\,\gamma_1 = \mathrm{dom}\,\gamma_2$, and using item (b) of Proposition 6 and (2) and (3) of Definitions 4 one deduces that we also have $(\boldsymbol{K}, F, \gamma_\iota) \in \mathcal{C}_0$ for ${}_{\iota=1,2}$. For every $s \in \mathrm{dom}\,\gamma_\iota$ we have $\gamma_\iota`s = h_\iota`(x,u,s)$, hence $(x,u,s,\gamma_\iota`s) \in h_\iota \subseteq h^0$, and consequently $f`(x+s\,u) = f`x + s\,(\gamma_\iota`s)$, whence for invertible s we get $\gamma_1`s = \iota`s\,(f`(x+s\,u) - f`x) = \gamma_2`s$. By (5) of Definitions 4 we get $\gamma_1 = \gamma_2$, whence finally $h_1`W = h_1`(\gamma`t) = h_1 \circ \gamma`t = \gamma_1`t = \gamma_2`t = h_2 \circ \gamma`t = h_2`(\gamma`t) = h_2`W$. $\square$

Our definition above of k^{th} order $\mathcal{C}_0-{}_{\mathrm{BGN}}$ differentiability over $\boldsymbol{K}$ precisely captures the content of [2; Definitions 2.1, 4.1, Remark 4.3, pp. 222, 228] because of the recursion rule given by the following

10 Proposition. *Let $\mathcal{C}_0$ be a* BGN *– class over $\boldsymbol{K}$. For all $\tilde{f}, k$ then*

$$\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K}) \Leftrightarrow \tilde{f} \in \mathcal{D}^{1.}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K}) \textit{ and } \mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} \in \mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K})\,.$$

Proof. Having the setting fixed, to simplify the notations, we agree to let $\mathcal{D}_k = \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K})$ and $\Delta\,\tilde{f} = \mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f}$ and

$$\mathrm{S}(\tilde{f}, k) = \{\,\boldsymbol{f} : (\emptyset, \tilde{f}) \in \boldsymbol{f} \in {\mathcal{C}_0}^{k+1.} \text{ and } \forall\, i \in k\,;\ \boldsymbol{f}`i^+ = \Delta(\boldsymbol{f}`i)\,\}\,.$$

Now letting $\tilde{f} \in \mathcal{D}_{k+1.}$, there is $\boldsymbol{f} \in \mathrm{S}(\tilde{f}, k+1.)$. With $\iota = \langle\, i^+ : i \in k+1.\rangle$, then $\boldsymbol{f}\,|\,2. \in \mathrm{S}(\tilde{f}, 1.)$ and $\boldsymbol{f} \circ \iota \in \mathrm{S}(\Delta\,\tilde{f}, k)$, whence we get $\tilde{f} \in \mathcal{D}_{1.}$ and $\Delta\,\tilde{f} \in \mathcal{D}_k$. Conversely, if $\tilde{f} \in \mathcal{D}_{1.}$ and $\Delta\,\tilde{f} \in \mathcal{D}_k$, there are $\boldsymbol{f}_0 \in \mathrm{S}(\tilde{f}, 1.)$ and $\boldsymbol{f}_1 \in \mathrm{S}(\Delta\,\tilde{f}, k)$. Then $\boldsymbol{f}_0 \cup (\boldsymbol{f}_1 \circ \iota^{-\iota}) \in \mathrm{S}(\tilde{f}, k+1.)$, whence we see that $\tilde{f} \in \mathcal{D}_{k+1.}$. $\square$

Note that even if the productive class $\mathcal{C}_0$ over $\boldsymbol{K}$ is not a BGN – one, we still have a recursion that $\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K})$ if and only if some first order BGN – difference quotient map $\tilde{f}^{1}$ for $\tilde{f}$ in $\mathcal{C}_0$ over $\boldsymbol{K}$ satisfies $\tilde{f}^{1} \in \mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K})$, put concisely

$$\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K}) \;\Leftrightarrow\; \tilde{f} \in \mathbf{U} \text{ and } \mathcal{C}_0\text{-}\mathrm{Difq}^{1}_{\boldsymbol{K}}``\{\tilde{f}\} \cap \mathcal{D}^{k}_{\mathrm{BGN}}(\mathcal{C}_0, \boldsymbol{K}) \neq \emptyset\,.$$

11 *Remark.* The transition from our reinterpretation of the BGN setting to the original one and vice versa is accomplished as follows. Given our $\mathcal{C}_0$, upon putting

$$\boldsymbol{\tau} = \langle\,\tau_{\mathrm{rd}}\,(\mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}`P) : P \in \mathrm{dom}\,\mathcal{C}_0\,\rangle_{\mathrm{old}} \qquad \text{and}$$

$$\mathcal{C}^0 = \langle\,\langle\,\mathcal{C}_0``\{P\} \cap \{\,f : \mathrm{dom}\,f = U\,\} : U \in \tau\sigma_{\mathrm{rd}}\,P\,\rangle_{\mathrm{old}} : P \in \mathrm{dom}\,\mathcal{C}_0\,\rangle_{\mathrm{old}}\,,$$

then $\mathcal{C}^0$ is the family associating with each pair $P = (E,F) \in \mathcal{O}^{\times 2.} = \mathrm{dom}\,\mathcal{C}_0$ the small family $\tau_{\mathrm{rd}}\,E \ni U \mapsto \mathcal{C}^0`P`U$ such that $\mathcal{C}^0`P`U$ is the set of functions which in [2; Definition 1.1(b), p. 219] is denoted by "$C^0(U,F)$", where thus the dependence on E is suppressed.

Likewise $\boldsymbol{\tau}$ maps the pair $Q = (E_1, E_2) \in \mathrm{dom}\,\mathcal{C}_0$ to the topology $\boldsymbol{\tau}`Q$ for the underlying set $(\upsilon_s E_1) \times (\upsilon_s E_2)$ of the algebraic product module $(\sigma_{\mathrm{rd}}\,E_1) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}}\,E_2)$ which in [2; Definition 1.1(c), p. 219] is denoted by "$\mathcal{T}(E_1 \times E_2)$".

Conversely, given only the BGN – family $\mathcal{C}^0$, we get

$$\mathcal{C}_0 = \{\,(P,f) : \exists\,\boldsymbol{c}\,;\ (P,\boldsymbol{c}) \in \mathcal{C}^0 \text{ and } f \in \boldsymbol{c}`(\mathrm{dom}\,f) \neq \mathbf{U}\,\}\,.$$

Hence $\boldsymbol{\tau}$ is not needed to get $\mathcal{C}_0$, only its existence is needed to get the properties of $\mathcal{C}_0\text{-}\mathrm{P}_{\mathrm{rod}}$. Note further that our $\mathcal{O}$ = the $\mathcal{M}$ in [2; Definition 1.1(a), p. 219].

The difference quotient map $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f}$ for $\tilde{f} = (E,F,f)$ with $\mathrm{dom}\,f = U$ in the case where we have that $\mathcal{C}_0\text{-}\bar{\Delta}_{\boldsymbol{K}}\,\tilde{f} \neq \mathbf{U}$ is the one determined by the mapping data $f^{[1]}$: "$U \times E \times \mathbb{K}$" $\supseteq U^{[1]} \to F$ in [2; Definition 2.1, p. 222] associated with the mapping data $f : E \supseteq U \to F$ corresponding to the map $\tilde{f}$.

B. Compactly generated topologies, vector and function spaces

Here we develop those properties of compactly generated real vector spaces which are needed in proving that Seip's higher order differentiability concepts can be obtained as a particular case of the general construction given above.

The most important particular result obtained of which we shall need to make explicit use several times in the sequel is Proposition 31 below giving a certain exponential law property of the space $\mathcal{L}_{\mathrm{cg}}(E,F)$ of continuous linear maps $E \to F$ topologized by the k-extension of the compact open topology.

First, to make matters precise, we begin by introducing the following

12 *Definitions.* Balancing between [13; p. 146] and the conventional [4; Definition 6.1, p. 237], we here agree on saying that a topology $\mathcal{T}$ is *locally compact* iff for all x, V with $x \in V \in \mathcal{T}$ there are K, U such that $x \in U \in \mathcal{T}$ and $U \subseteq K \subseteq V$ and K is $\mathcal{T}$–compact in the "non-Bourbakian" sense [13; p. 135].

Our *Kelleyfication* of a topology $\mathcal{T}$ is

$$k_{\mathrm{elt}}\mathcal{T} = \{\, U : U \subseteq \bigcup \mathcal{T} \text{ and } \forall K\,;\ \exists V\,; \\ K \text{ is } \mathcal{T}\text{–compact} \Rightarrow V \in \mathcal{T} \text{ and } U \cap K = V \cap K \,\}\,.$$

Our definition of the *k-extension* $k_{\mathrm{ex}}\mathcal{T}$ of $\mathcal{T}$ is obtained from the preceding by putting there "K is $\mathcal{T}$–closed compact" in place of "K is $\mathcal{T}$–compact".

A topology $\mathcal{T}$, and the topological space $(\bigcup \mathcal{T}, \mathcal{T})$ we say to be *almost compactly generated* iff also $\mathcal{T} = k_{\mathrm{elt}}\mathcal{T}$ holds, and we say *compactly generated* iff also $\mathcal{T}$ is Hausdorff with $\mathcal{T} = k_{\mathrm{elt}}\mathcal{T}$. We let $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{U} = k_{\mathrm{elt}}(\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U})$, and also put $\kappa_{\mathrm{tv}} = \{\, (X, \mathcal{T}\,; X, k_{\mathrm{elt}}\mathcal{T}) : X = X \text{ and } \mathcal{T} \text{ is a topology} \,\}$.

Observe that according to the preceding definitions *every locally compact topology is almost compactly generated*, and every Hausdorff locally compact topology is compactly generated. In view of [13; Theorem 5.17, p. 146], *every compact topology which is also Hausdorff or regular is locally compact.*

Since $U \in \mathcal{T}$ whenever $\mathcal{T}$ is a first countable topology and U is sequentially $\mathcal{T}$–open, it follows, cf. [13; Theorem 7.13, p. 231], that *first countable topologies are almost compactly generated* and *metrizable topologies are compactly generated*.

Note the following slight difference between our definitions above and [13; pp. 230–231]. A topological space $(\Omega, \mathcal{T})$ is a *k-space* in the sense of [13] iff we have $\mathcal{T} = k_{\mathrm{ex}}\mathcal{T}$, and this implies that $\mathcal{T}$ is almost compactly generated since we generally have $k_{\mathrm{elt}}\mathcal{T} \subseteq k_{\mathrm{ex}}\mathcal{T}$, equality here holding if also $\mathcal{T}$ is Hausdorff.

13 *Example.* We assume that $[0,1] \subseteq \Omega \subseteq I\!\!R$, and we put $A = \Omega \setminus \,]0,1[$ and $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$ where we have

$$\mathcal{B} = \big\{\,]s,t[\, : 0 < s < t < 1 \big\} \cup \big\{\,]s,1[\, \cup \{t\} : 0 < s < 1 \text{ and } t \in A \,\big\}\,.$$

Then $\mathcal{T}$ is a first countable locally compact topology for Ω which is neither Hausdorff nor regular. Hence $\mathcal{T}$ is almost compactly generated, but if the set A is infinite, then $(\Omega, \mathcal{T})$ is not a k-space. Indeed, then $k_{\mathrm{elt}}\mathcal{T} \neq k_{\mathrm{ex}}\mathcal{T}$ holds since we have $\{t\} \in k_{\mathrm{ex}}\mathcal{T} \setminus k_{\mathrm{elt}}\mathcal{T}$ for every $t \in A$, noting that for every $\mathcal{T}$–closed compact set K there is s with $0 < s < 1$ and $]s,1[\, \cap K = \emptyset$.

14 *Definitions.* The class $\mathrm{cgVS}(\boldsymbol{R})$ of real *compactly generated vector spaces* we let have as its members exactly the real topologized vector spaces $E = (X, \mathcal{T})$ such that $\mathcal{T}$ is compactly generated and $\sigma_{\mathrm{rd}}X$ is continuous $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{T} \to \mathcal{T}$ and $\tau_{\mathrm{rd}}X$ is continuous $\tau_{I\!\!R} \underset{\mathrm{t}}{\times} \mathcal{T} \to \mathcal{T}$. The class $\mathrm{scgVS}(\boldsymbol{R})$ of *Seip–convenient* spaces has as its members exactly the $E \in \mathrm{cgVS}(\boldsymbol{R})$ such that $\mathcal{U}$ exists with $(\sigma_{\mathrm{rd}}E, \mathcal{U})$ a

sequentially complete Hausdorff locally convex space and $\tau_{\mathrm{rd}} E = k_{\mathrm{elt}} \mathcal{U}$. We let $E \mathbin{\boxed{\scriptstyle\mathrm{k}}} F = ((\sigma_{\mathrm{rd}} E) \underset{\mathrm{vs}}{\times} (\sigma_{\mathrm{rd}} F)), (\tau_{\mathrm{rd}} E) \underset{\mathrm{k}}{\times} (\tau_{\mathrm{rd}} F))$. With every $\boldsymbol{E} \in \mathrm{cgVS}(\boldsymbol{R})^{I}$ and any set I we associate the compactly generated product vector space

$$\prod\nolimits_{\mathrm{cgVS}\,\mathbb{R}} \boldsymbol{E} = (\prod\nolimits_{\mathrm{VS}\,\mathbb{R}} (\mathrm{pr}_1 \circ \boldsymbol{E}), k_{\mathrm{elt}} \prod\nolimits_{\mathbf{ti}} (\mathrm{pr}_2 \circ \boldsymbol{E})).$$

We also let $\tau_{\mathrm{kv}} = \langle \leq_{\mathrm{LCS}}(\boldsymbol{R})\text{-}\sup(\kappa_{\mathrm{tv}}{}^{-\iota}``\{F\} \cap \mathrm{LCS}(\boldsymbol{R})) : F \in \mathrm{cgVS}(\boldsymbol{R}) \rangle$.

Note that for $F \in \mathrm{scgVS}(\boldsymbol{R})$ the class $\kappa_{\mathrm{tv}}{}^{-\iota}``\{F\} \cap \mathrm{LCS}(\boldsymbol{R})$ need not be a singleton. E.g., if with $E = \coprod_{\mathrm{LCS}\,\mathbb{R}} (\mathbb{R} \times \{\boldsymbol{R}\})$ we let $E_1 = (\sigma_{\mathrm{rd}} E, \mathcal{T}_{\mathrm{box}})$ where $\mathcal{T}_{\mathrm{box}}$ is the "box topology" having $\left\{ v_s E \cap \prod_{\mathbf{c}} \boldsymbol{V} : \boldsymbol{V} \in \{]-n^{-1}, n^{-1}[: n \in \mathbb{Z}^{+} \}^{\mathbb{R}} \right\}$ as a filter basis for $\mathcal{N}_o E_1 = \mathcal{N}_{bh}(\mathbb{R} \times \{0\}, \mathcal{T}_{\mathrm{box}})$, then $E \neq E_1$ with $\mathcal{B}_s E = \mathcal{B}_s E_1$ whence compactness is the same for $\tau_{\mathrm{rd}} E$ and $\tau_{\mathrm{rd}} E_1$, and so $\kappa_{\mathrm{tv}}` E = \kappa_{\mathrm{tv}}` E_1$.

When $F \in \mathrm{scgVS}(\boldsymbol{R})$, the space $\tau_{\mathrm{kv}}` F$ is the strictest locally convex space looser than $F$. Put otherwise, the topology $\tau_{\mathrm{rd}}(\tau_{\mathrm{kv}}` F)$ is the finest locally convex topology on the vector space $\sigma_{\mathrm{rd}} F$ which is coarser than $\tau_{\mathrm{rd}} F$. In [16; p. 38 ff.], the space $\tau_{\mathrm{kv}}` F$ is denoted by "$\mathrm{LK}(F)$", and it is sequentially complete, noting that if $E, E_1 \in \kappa_{\mathrm{tv}}{}^{-\iota}``\{F\} \cap \mathrm{LCS}(\boldsymbol{R})$ with E sequentially complete and $E \leq E_1$, then also E_1 is sequentially complete.

15 Lemma. *If $(\mathcal{T}, \mathcal{U}, f)$ is a global topological map, so is $(k_{\mathrm{elt}} \mathcal{T}, k_{\mathrm{elt}} \mathcal{U}, f)$.*

Proof. Under the premise, if $V \in k_{\mathrm{elt}} \mathcal{U}$ and K is $\mathcal{T}-$compact, we have to prove that there is $U \in \mathcal{T}$ with $f^{-\iota}`` V \cap K = U \cap K$. For $K_1 = f`` K$ now K_1 is $\mathcal{U}-$compact, whence there is $V_1 \in \mathcal{U}$ with $V_1 \cap K_1 = V \cap K_1$. Putting $U = f^{-\iota}`` V_1$, we have $U \in \mathcal{T}$, and also $f^{-\iota}`` V \cap K = U \cap K$ holds. □

16 Lemma. *If $\mathcal{T}$ and $\mathcal{U}$ are any topologies, then $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{U} = (k_{\mathrm{elt}} \mathcal{T}) \underset{\mathrm{k}}{\times} (k_{\mathrm{elt}} \mathcal{U})$. For any small family $\boldsymbol{\mathcal{T}}$ of topologies it holds that*

$$k_{\mathrm{elt}} \prod\nolimits_{\mathbf{ti}} \boldsymbol{\mathcal{T}} = k_{\mathrm{elt}} \prod\nolimits_{\mathbf{ti}} \{ (i, k_{\mathrm{elt}} \mathcal{T}) : (i, \mathcal{T}) \in \boldsymbol{\mathcal{T}} \}.$$

Proof. Let $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{U} = \mathcal{T}_2$ be topologies, and put $\mathcal{W} = (k_{\mathrm{elt}} \mathcal{T}) \underset{\mathrm{k}}{\times} (k_{\mathrm{elt}} \mathcal{U})$ and $P = \bigcup \mathcal{W}$. Since $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{U} = k_{\mathrm{elt}}(\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U})$ and $\mathcal{W} = k_{\mathrm{elt}}((k_{\mathrm{elt}} \mathcal{T}) \underset{\mathrm{t}}{\times} (k_{\mathrm{elt}} \mathcal{U}))$, by Lemma 15 above we trivially have $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{U} \subseteq \mathcal{W}$. To get the converse, it suffices that $\mathrm{id}\, P$ is continuous $\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{U} \to \mathcal{W}$. This in turn follows if for arbitrarily given $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U}-$compact K and $p \in K$, we have that $\mathrm{id}\, K$ is continuous $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U} \to \mathcal{W}$ at the point p. That is, given $p \in W \in \mathcal{W}$, there should (x) exist some $U \in \mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U}$ such that $p \in U$ and $U \cap K \subseteq W$ hold.

To obtain (x) above, letting $B_1 = \mathrm{dom}\, K$ and $B_2 = \mathrm{rng}\, K$ and $C = B_1 \times B_2$, for ${}_{\iota = 1,2}$ then B_ι is $\mathcal{T}_\iota-$compact, hence $k_{\mathrm{elt}} \mathcal{T}_\iota-$compact, and consequently C is $k_{\mathrm{elt}} \mathcal{T} \underset{\mathrm{t}}{\times} k_{\mathrm{elt}} \mathcal{U}-$compact. Hence, there is $W_1 \in k_{\mathrm{elt}} \mathcal{T} \underset{\mathrm{t}}{\times} k_{\mathrm{elt}} \mathcal{U}$ with $p \in W \cap C = W_1 \cap C$, whence further there are $V_\iota \in k_{\mathrm{elt}} \mathcal{T}_\iota$ with $p \in V_1 \times V_2 \subseteq W_1$. Then there are $U_\iota \in \mathcal{T}_\iota$ with $V_\iota \cap B_\iota = U_\iota \cap B_\iota$. Putting $U = U_1 \times U_2$, we have $U \in \mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{U}$ with $p \in V_1 \times V_2 \cap C = U_1 \times U_2 \cap C \subseteq U$ and $U \cap K = U \cap C \cap K \subseteq V_1 \times V_2 \cap C \subseteq W_1 \cap C = W \cap C \subseteq W$, hence (x) as we wished.

We leave it as an exercise to the reader to naturally generalize the preceding idea in order to establish the latter assertion. As a hint we only mention that with the family $\boldsymbol{K} = \{ (i, S) : \emptyset \neq S = \{ \xi : \exists\, x\, ; (i, \xi) \in x \in K \} \}$ one now considers $C = \prod_{\mathbf{c}} \boldsymbol{K}$, and one finally obtains $U = \prod_{\mathbf{c}} \boldsymbol{U}$ for some $\boldsymbol{U} \in \prod_{\mathbf{c}} \boldsymbol{\mathcal{T}}$ with the property that $\{ i : \boldsymbol{U}` i \neq \bigcup(\boldsymbol{\mathcal{T}}` i) \}$ is a finite set. □

17 Lemma. *Let $\mathcal{T}$ be a compactly generated topology, and let $A \subseteq \Omega = \bigcup \mathcal{T}$. If in addition $A \in \mathcal{T}$ or $\Omega \setminus A \in \mathcal{T}$, then $\mathcal{T} \cap A$ is a compactly generated topology.*

Proof. First letting $\Omega \setminus A \in \mathcal{T}$, arbitrarily given $B \subseteq A$ such that for every $\mathcal{T} \cap A-$compact K_1 there is $U \in \mathcal{T}$ with $U \cap K_1 = B \cap K_1$, as for $V = \Omega \setminus A \cup B$ we have $V \cap A = B$, it suffices that $V \in \mathcal{T}$. As $\mathcal{T}$ is compactly generated, for this it suffices that for each given $\mathcal{T}-$compact K there is $U \in \mathcal{T}$ with $U \cap K = V \cap K$. Putting $K_1 = K \cap A$, now K_1 is $\mathcal{T} \cap A-$compact, whence there is $U_1 \in \mathcal{T}$ with $U_1 \cap K_1 = B \cap K_1$, hence $U_1 \cap K \cap A = B \cap K$. For $U = \Omega \setminus A \cup U_1$ then $U \in \mathcal{T}$ and $U \cap K = \Omega \setminus A \cup U_1 \cap K = \Omega \setminus A \cup B \cap K = V \cap K$.

Next assuming that $A \in \mathcal{T}$, arbitrarily given $B \subseteq A$ such that for every $\mathcal{T} \cap A-$compact K_1 there is $U \in \mathcal{T}$ with $U \cap K_1 = B \cap K_1$, for arbitrarily fixed $\mathcal{T}-$compact K we must prove (k) that there is $U \in \mathcal{T}$ with $U \cap K = B \cap K$. For this considering $\mathcal{U} = \mathcal{T} \cap \{\, U : \exists\, x\,;\, x \in U \cap K \subseteq B \cap K \,\}$, if for arbitrarily fixed $x \in B \cap K$ we prove (x) that there is $U \in \mathcal{T}$ with $x \in U \cap K \subseteq B \cap K$, we are done since to get (k) we may take $U = \bigcup \mathcal{U}$.

To establish (x), putting $K_2 = K \setminus A$, then K_2 is $\mathcal{T}-$compact whence noting that $\mathcal{T}$ is Hausdorff, there are disjoint $U_1, U_2 \in \mathcal{T}$ with $x \in U_1$ and $K_2 \subseteq U_2$. Putting $K_1 = K \setminus U_2$, now K_1 is $\mathcal{T} \cap A-$compact whence there is $U_0 \in \mathcal{T}$ with $U_0 \cap K_1 = B \cap K_1$. Taking $U = U_0 \cap U_1$, we have obtained (x) since by $U_1 \cap U_2 = \emptyset$ we have

$$
\begin{aligned}
x &\in B \cap K \cap U_1 = B \cap K \cap U_1 \setminus U_2 = B \cap K_1 \cap U_1 = U_0 \cap K_1 \cap U_1 \\
&= U \cap (K \setminus U_2) \subseteq U \cap K \subseteq U_0 \cap U_1 \cap K = U_0 \cap U_1 \cap K \setminus U_2 \\
&= U_0 \cap U_1 \cap K_1 = B \cap K_1 \cap U_1 \subseteq B \cap K_1 \subseteq B \cap K. \qquad \square
\end{aligned}
$$

18 Corollary. *Let $E = (X, \mathcal{T}) \in \mathrm{cgVS}(\boldsymbol{R})$ and $F = E_{/S}$ where S is a vector subspace in X which is $\mathcal{T}-$closed. Then $F \in \mathrm{cgVS}(\boldsymbol{R})$. If in addition E is Seip$-$convenient, so is F too.*

Proof. To get $F \in \mathrm{cgVS}(\boldsymbol{R})$, noting that by Lemma 17 we have $\tau_{\mathrm{rd}} F = \mathcal{T} \cap S$ compactly generated, it suffices (a) that $\sigma^2_{\mathrm{rd}} F$ is continuous $\tau_{\mathrm{rd}} F \underset{\mathrm{k}}{\times} \tau_{\mathrm{rd}} F \to \mathcal{T}$, and (b) that $\tau\sigma_{\mathrm{rd}} F = \tau_{\mathrm{rd}} X \,|\, (I\!R \times S)$ is continuous $\tau_{I\!R} \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F \to \mathcal{T}$. We immediately get (b) from continuity of $\tau_{\mathrm{rd}} X : \tau_{I\!R} \underset{\mathrm{t}}{\times} \mathcal{T} \to \mathcal{T}$, and we deduce (a) as follows. Letting $\iota = \mathrm{id}\,(S^{\times 2 \cdot})$, since $\sigma^2_{\mathrm{rd}} F = \sigma_{\mathrm{rd}} X \circ \iota$, it suffices that ι is continuous $\tau_{\mathrm{rd}} F \underset{\mathrm{k}}{\times} \tau_{\mathrm{rd}} F \to \mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{T}$, and by Lemma 15 this follows if ι is continuous $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{T} \cap (S^{\times 2 \cdot}) = \tau_{\mathrm{rd}} F \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F \to \mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{T}$. This is trivial.

Assuming that E is Seip$-$convenient, there is $\mathcal{U}$ such that $(X, \mathcal{U}) \in \mathrm{LCS}(\boldsymbol{R})$ is sequentially complete with $\mathcal{T} = k_{\mathrm{elt}} \mathcal{U}$. Then also $(\sigma_{\mathrm{rd}} F, \mathcal{U} \cap S) \in \mathrm{LCS}(\boldsymbol{R})$, and is sequentially complete. To prove that $\mathcal{T} \cap S = k_{\mathrm{elt}}(\mathcal{U} \cap S)$, noting that we have $k_{\mathrm{elt}}(\mathcal{U} \cap S) \subseteq \mathcal{T} \cap S$ by Lemma 15 since $\mathcal{T} \cap S$ is compactly generated, it suffices that $\mathrm{id}\, S$ is continuous $k_{\mathrm{elt}}(\mathcal{U} \cap S) \to \mathcal{T}$, but this is immediate by Lemma 15 from continuity of $\mathrm{id}\, S : \mathcal{U} \cap S \to \mathcal{U}$. $\square$

For sets of functions we obtain topologies, and topologized linear structures with the aid of the following basic

19 *Constructions.*

(1) $\mathrm{bT_{op}}\, \mathcal{B} = \{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \{ \bigcap \mathcal{B}_1 : \mathcal{B}_1 \subseteq \mathcal{B}$ and $\mathcal{B}_1$ is finite $\}\}$,

(2) $\mathrm{tF_{so}}\, S_{\mathcal{G}} \mathcal{T} = \mathrm{bT_{op}} \{\, S \cap \{\, x : x^{``} B \subseteq V \,\} : B \in \mathcal{G}$ and $V \in \mathcal{T} \,\}$,

(3) $\mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U}) = \{\, U : \forall\, C, \mathcal{K}\,;\, C = (\bigcup \mathcal{U})^{\bigcup \mathcal{T}} \cap \{\, x : x^{-\iota ```} \mathcal{U} \subseteq \mathcal{T} \,\}$
and $\mathcal{K} = \{\, K : K$ is $\mathcal{T}-$compact $\} \Rightarrow U \in \mathrm{tF_{so}}\, C_{\mathcal{K}} \mathcal{U} \,\}$,

(4) $\mathrm{vF_{so}}\, S_{\mathcal{G}} F = \big((\sigma_{\mathrm{rd}} F)^{\mathrm{dom} \bigcup S]_{\mathrm{vs}}}{}_{|\, S} \,,\, \mathrm{tF_{so}}\, S_{\mathcal{G}} \tau_{\mathrm{rd}} F \big)$,

(5) $\mathrm{vF_{su}}\, S_{\mathcal{G}}\, F = \bigcap \{ ((\sigma_{\mathrm{rd}}\, F)^{\,\Omega]_{\mathrm{vs}}}{}_{|\,S}\, , \mathcal{T}\,) : F, S : \Omega = \mathrm{dom} \bigcup S \text{ and}$
$\mathcal{T} = \{\, U : U \subseteq S \text{ and } \forall\, x \in U\, ;\, \exists\, B \in \mathcal{G}\, , V \in \mathcal{N}_o\, F\, ;$
$\forall\, u \in S\, ;\, u``B \subseteq V \Rightarrow \sigma_{\mathrm{rd}}^{2}\, F \circ [\, x\, , u\,] \in U \,\}\}\, .$

One immediately observes that $\mathrm{bT_{op}}\, \mathcal{B}$ is a topology for $\bigcup \mathcal{B}$ whenever $\mathcal{B} \in \mathbf{U}$. Consequently $\mathrm{tF_{so}}\, S_{\mathcal{G}}\, \mathcal{T}$ is a topology whenever S is a set. However, it may be quite pathological or uninteresting, unless $\mathcal{T}$ is a topology and $\bigcup \mathcal{G} \subseteq \Omega$ and $S \subseteq (\bigcup \mathcal{T}\,)^{\,\Omega}$ for some Ω, and possibly some further conditions hold.

In particular, if $\mathcal{T}$ and $\mathcal{U}$ are topologies, then $\mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U}\,)$ is the *compact open* topology for the set $\bigcup \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U}\,)$ of functions x with $(\mathcal{T}, \mathcal{U}\, , x)$ a global topological map. Informally $\mathrm{vF_{su}}\, S_{\mathcal{G}}\, F$ may be called the set S of functions into F turned into the space with topology that of the uniform convergence on the members of $\mathcal{G}$.

20 Proposition. *Let* $F \in \mathrm{LCS}(\boldsymbol{R})$ *and* $\Omega = \bigcup \mathcal{G} \in \mathbf{U}$ *and* $X = (\sigma_{\mathrm{rd}}\, F)^{\,\Omega]_{\mathrm{vs}}}$. *Let* S *be a vector subspace in* X *such that* $\{\, x``B : x \in S \text{ and } B \in \mathcal{G}\,\} \subseteq \mathcal{B}_s\, F$, *and such that for any* $A\, , B \in \mathcal{G}$ *there is* $C \in \mathcal{G}$ *with* $A \cup B \subseteq C$. *Let* $E = \mathrm{vF_{su}}\, S_{\mathcal{G}}\, F$. *Then* $E \in \mathrm{LCS}(\boldsymbol{R})$ *with* $\sigma_{\mathrm{rd}}\, E = X_{|\,S}$. *If* $\mathcal{V}$ *is any filter basis for* $\mathcal{N}_o\, F$, *then the class* $\{\, S \cap \{\, x : x``B \subseteq V\,\} : B \in \mathcal{G} \text{ and } V \in \mathcal{V}\,\}$ *is a filter basis for* $\mathcal{N}_o\, E$.

Proof. See [12; Proposition 2.10.1, pp. 43–44]. □

Adapting the proof of [13; Theorem 7.11, p. 230], we obtain

21 Proposition. *Let* $F \in \mathrm{LCS}(\boldsymbol{R})$, *and let* $(\Omega\, , \mathcal{T}\,)$ *be any topological space with* $\mathcal{K} = \{\, K : K \text{ is } \mathcal{T}-\text{compact}\,\}$. *Also let* $\mathcal{F} = \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \tau_{\mathrm{rd}}\, F\,)$, *and let* S *be a vector subspace in* $(\sigma_{\mathrm{rd}}\, F)^{\,\Omega]_{\mathrm{vs}}}$ *with* $S \subseteq \bigcup \mathcal{F}$. *Then* $\tau_{\mathrm{rd}}\, \mathrm{vF_{su}}\, S_{\mathcal{K}}\, F = \mathcal{F} \cap S$.

Proof. Let $\mathcal{U} = \tau_{\mathrm{rd}}\, \mathrm{vF_{su}}\, S_{\mathcal{K}}\, F$ and $\mathcal{P}_2 = \mathcal{K} \times \tau_{\mathrm{rd}}\, F$. Also putting $N(x\, ; K, V\,) = \{\, \sigma_{\mathrm{rd}}^{2}\, F \circ [\, x\, , u\,] : u \in S \text{ and } u``K \subseteq V\,\}$, then by Proposition 20 above, for $x \in S$ the class $\{\, N(x\, ; K, V\,) : K \in \mathcal{K} \text{ and } V \in \mathcal{N}_o\, F\,\}$ is a filter basis for $\mathcal{N}_{bh}(x\, , \mathcal{U}\,)$.

To prove that $\mathcal{U} \subseteq \mathcal{F} \cap S$, arbitrarily given $x \in U \in \mathcal{U}$, there are some $K \in \mathcal{K}$ and $V \in \mathcal{N}_o\, F$ with $N(x\, ; K, V\,) \subseteq U$, and we should establish (x) existence of a finite $\mathcal{P} \subseteq \mathcal{P}_2$ such that for $N = S \cap \{\, y : \forall\, P \in \mathcal{P}\, ;\, y``\sigma_{\mathrm{rd}}\, P \subseteq \tau_{\mathrm{rd}}\, P\,\}$ we have $x \in N \subseteq N(x\, ; K, V\,)$. Let $\tilde{x} = (\mathcal{T}, \tau_{\mathrm{rd}}\, F, x)$.

To get (x), we first take a $\tau_{\mathrm{rd}}\, F-$closed $V_1 \in \mathcal{N}_o\, F$ with $V_1 - V_1 - V_1 \subseteq V$, and put $W = \{\, (\zeta\, , \xi) : \zeta - \xi \in V_1\,\}$. Now $x``K$ is $\tau_{\mathrm{rd}}\, F-$compact by continuity of $\tilde{x}$ and $\mathcal{T}-$compactness of $K$, whence there is a finite $A \subseteq K$ with $x``K \subseteq W \circ x``A$. Taking $\mathcal{P} = \{\, (K \cap (x^{-\iota} \circ W \circ x``\{Q\})\, , \mathrm{Int}_{\,\tau_{\mathrm{rd}}\, F}\, (W \circ W \circ x``\{Q\})) : Q \in A\,\}$, we have $\mathcal{P}$ finite with $\mathcal{P} \subseteq \mathcal{P}_2$ since by continuity of $\tilde{x}$ and $\tau_{\mathrm{rd}}\, F-$closedness of $V_1$ the set $x^{-\iota} \circ W \circ x``\{P\}$ is $\mathcal{T}-$closed for every $P \in A$. To see that $x \in N$, just note that if $P \in K \cap (x^{-\iota} \circ W \circ x``\{Q\})$ with $Q \in A$, we then have

$$x`P \in x\,[\, x^{-\iota} \circ W \circ x``\{Q\}\,] \subseteq W \circ x``\{Q\} \subseteq \mathrm{Int}_{\,\tau_{\mathrm{rd}}\, F}\, (W \circ W \circ x``\{Q\})\, .$$

To prove that $N \subseteq N(x\, ; K, V\,)$, arbitrarily fixing $y \in N$ and $P \in K$, it suffices to show that $(y - x)`P \in V$. Now, by $P \in K \subseteq x^{-\iota} \circ W \circ x``A$, there is $Q \in A$ with $P \in x^{-\iota} \circ W \circ x``\{Q\}$, consequently $P \in K \cap (x^{-\iota} \circ W \circ x``\{Q\})$. Letting

$$P = (K \cap (x^{-\iota} \circ W \circ x``\{Q\})\, , \mathrm{Int}_{\,\tau_{\mathrm{rd}}\, F}\, (W \circ W \circ x``\{Q\}))\, ,$$

then $P \in \mathcal{P}$ holds, and since by $y \in N$ we have $y``\sigma_{\mathrm{rd}}\, P \subseteq \tau_{\mathrm{rd}}\, P$, we get $y`P \in \mathrm{Int}_{\,\tau_{\mathrm{rd}}\, F}\, (W \circ W \circ x``\{Q\}) \subseteq W \circ W \circ x``\{Q\}$. We also have $x`P \in W \circ x``\{Q\}$, and hence $x`Q \in W^{-\iota} \circ x``\{P\}$. Consequently, we get $(x`P, y`P) \in W \circ W \circ W^{-\iota}$, that is $(y - x)`P = y`P - x`P \in V_1 - V_1 - V_1 \subseteq V$.

Conversely, to prove that $\mathcal{F} \cap S \subseteq \mathcal{U}$, arbitrarily given a nonempty $K \in \mathcal{K}$ and $U \in \tau_{\mathrm{rd}}\, F$ and $x \in S$ with $C = x``K \subseteq U$, it suffices to find $V \in \mathcal{N}_o\, F$ with

$N(x\,;K,V\,) \subseteq S \cap \{\, y : y``K \subseteq U\,\}$. Noting that this immediately follows if we have $C+V \subseteq U$, it suffices to establish the latter. For this, we consider $\mathcal{N}_1 = C \times \mathcal{N}_o F \cap \{\,(\xi\,,V\,) : \xi + V + V \subseteq U\,\}$. Since $(\sigma^2_{\mathrm{rd}} F, \tau_{\mathrm{rd}} F)$ is a topological group, we have $C \subseteq \operatorname{dom} \mathcal{N}_1$, and since C is $\tau_{\mathrm{rd}} F-$compact, a finite $\mathcal{N} \subseteq \mathcal{N}_1$ exists with $C \subseteq \{\,\xi + \zeta : \exists\, V\,;\,(\xi\,,V\,) \in \mathcal{N} \text{ and } \zeta \in V\,\}$. Taking now $V = \bigcap \operatorname{rng} \mathcal{N}$, we have $V \in \mathcal{N}_o F$ with $C + V \subseteq \{\,\xi + \zeta + \zeta_1 : \exists\, V\,;\,(\xi\,,V\,) \in \mathcal{N} \text{ and } \zeta\,,\zeta_1 \in V\,\} \subseteq U$. $\square$

Observe above that when establishing the inclusions in both directions between $\mathcal{U}$ and $\mathcal{F} \cap S$, we needed to know that $(\sigma^2_{\mathrm{rd}} F, \tau_{\mathrm{rd}} F)$ is a topological group. The preceding proof hence does not give a corresponding result if we assume F to be a compactly generated vector space instead of a topological one. In fact, if the topologized linear structure of $F \in \mathrm{cgVS}(\boldsymbol{R})$ determines a uniformity in the natural manner, then F necessarily is a topological vector space.

22 Proposition. *Let $\mathcal{T}$ and $\mathcal{U}$ be any topologies such that* (a) *or* (b) *below holds. Then* $\mathrm{ev} \,|\, \bigcup \mathcal{W}$ *is continuous* $\mathcal{W} \to \mathcal{U}$.

(a) *$\mathcal{T}$ is locally compact and* $\mathcal{W} = \mathcal{T} \underset{\mathrm{t}}{\times} \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U})$,

(b) *$\mathcal{T}$ is Hausdorff and* $\mathcal{W} = \mathcal{T} \underset{\mathrm{k}}{\times} \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U})$.

Proof. We give the deduction assuming (b) leaving it as an exercise to the reader to extract therefrom the almost trivial case when (a) is assumed instead. Putting $\mathcal{F} = \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{U})$, arbitrarily given a $\mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{F}-$compact K_2 and $V \in \mathcal{U}$, there should be some $W \in \mathcal{T} \underset{\mathrm{t}}{\times} \mathcal{F}$ with $\mathrm{ev}^{-\iota}``V \cap K_2 = W \cap K_2$. Supposing we have (x) that for any $z \in \mathrm{ev}^{-\iota}``V \cap K_2$ there are $N \in \mathcal{T}$ and $U \in \mathcal{F}$ with $z \in N \times U$ and $\mathrm{ev}\,[\, N \times U \cap K_2\,] \subseteq V$, we are done since $W = \bigcup \mathcal{A}$ will do for

$$\mathcal{A} = \{\, N \times U : (N, U) \in \mathcal{T} \times \mathcal{F} \text{ and } \mathrm{ev}\,[\, N \times U \cap K_2\,] \subseteq V\,\}.$$

To get (x) we arbitrarily fix $z = (P, x) \in \mathrm{ev}^{-\iota}``V \cap K_2$, and first note that by continuity of $x : \mathcal{T} \to \mathcal{U}$ there is $N_0$ with $P \in N_0 \in \mathcal{T}$ and $x``N_0 \subseteq V$. For $K_0 = \operatorname{dom} K_2$ and $K_1 = K_0 \setminus N_0$, then K_0 and K_1 are $\mathcal{T}-$compact. Now considering $\mathcal{N}_1 = \mathcal{T}^{\times 2\cdot} \cap \{\,(N_1, N_2) : \exists\, Q\,;\,(P, Q) \in N_1 \times N_2 \text{ and } N_1 \cap N_2 = \emptyset\,\}$, since $\mathcal{T}$ is Hausdorff and $P \notin K_1$, we have $K_1 \subseteq \bigcup \operatorname{rng} \mathcal{N}_1$, whence by compactness there is a finite $\mathcal{N} \subseteq \mathcal{N}_1$ with $K_1 \subseteq \bigcup \operatorname{rng} \mathcal{N}$. Taking $N = \bigcap \operatorname{dom} \mathcal{N}$ and $K = K_0 \setminus \bigcup \operatorname{rng} \mathcal{N}$ and $U = \bigcup \mathcal{F} \cap \{\, u : u``K \subseteq V\,\}$, then $P \in N \in \mathcal{T}$ and K is $\mathcal{T}-$compact with $N \cap K_0 \subseteq K \subseteq N_0$, and consequently $U \in \mathcal{F}$ with $z \in N \times U$. In addition, we also have $\mathrm{ev}\,[\, N \times U \cap K_2\,] \subseteq V$ since if $z = (Q\,, u) \in N \times U \cap K_2$, then we get $Q \in N \cap \operatorname{dom} K_2 = N \cap K_0 \subseteq K$ and $u``K \subseteq V$, whence $\mathrm{ev}`z = u`Q \in V$. $\square$

The preceding might be compared to one half of [16; Lemma 1.13, p. 13].

23 Proposition. *Let* $(\Omega\,, \mathcal{U}\,), (T, \mathcal{T}\,), (T_1, \mathcal{T}_1)$ *be any topological spaces such that* (a) *or* (b) *below holds, and let* $f \in {T_1}^{\Omega \times T}$. *Then f is continuous $\mathcal{W} \to \mathcal{T}_1$ if and only if $f^{\vee}$ is continuous* $\mathcal{U} \to \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{T}_1)$.

(a) *$\mathcal{T}$ is locally compact and* $\mathcal{W} = \mathcal{U} \underset{\mathrm{t}}{\times} \mathcal{T}$,

(b) *$\mathcal{T}$ is Hausdorff and $k_{\mathrm{elt}}\mathcal{U} = \mathcal{U}$ and* $\mathcal{W} = \mathcal{U} \underset{\mathrm{k}}{\times} \mathcal{T}$.

Proof. Again leaving to the reader the simpler case where (a) holds, under the assumption (b) putting $\mathcal{F} = \mathrm{t}C_{\mathrm{co}}(\mathcal{T}, \mathcal{T}_1)$, we first assume that f is continuous $\mathcal{W} \to \mathcal{T}_1$, and proceed to prove that $f^{\vee}$ is continuous $\mathcal{U} \to \mathcal{F}$ as follows. Arbitrarily given a $\mathcal{U}-$compact K, it suffices that $f^{\vee} | K$ is continuous $\mathcal{U} \cap K \to \mathcal{F}$. For this, arbitrarily fixing $P \in K$, and a $\mathcal{T}-$compact B and $V \in \mathcal{T}_1$ such that

$f\,[\,\{P\}\times B\,]\subseteq V$, it suffices to find $N\in\mathcal{N}_{bh}(P,\mathcal{U}\cap K)$ with $f\,[\,N\times B\,]\subseteq V$. To establish this, we consider

$$\mathcal{N}_1=\{\,(N,U):\exists\,x\,;\,N\in\mathcal{N}_{bh}(P,\mathcal{U}\cap K)\text{ and }\\ U\in\mathcal{N}_{bh}(x,\mathcal{T}\cap B)\text{ and }f\,[\,N\times U\,]\subseteq V\,\}\,.$$

Letting $K_2=K\times B$, as K_2 is $\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T}-$compact, by continuity of $f:\mathcal{W}\to\mathcal{T}_1$, we obtain continuity of $f\,|\,K_2:(\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T})\cap K_2\to\mathcal{T}_1$. Using this, we see that $B\subseteq\bigcup\operatorname{rng}\mathcal{N}_1$, whence by compactness of B there is a finite $\mathcal{N}\subseteq\mathcal{N}_1$ with $B\subseteq\bigcup\operatorname{rng}\mathcal{N}$. Taking $N=\bigcap\operatorname{dom}\mathcal{N}$, we have $N\in\mathcal{N}_{bh}(P,\mathcal{U}\cap K)$, and one quickly checks that also $f\,[\,N\times B\,]\subseteq V$ holds.

Conversely, next assuming that $f^{\vee}$ is continuous $\mathcal{U}\to\mathcal{F}$, we get continuity of $f:\mathcal{W}\to\mathcal{T}_1$ as follows. Putting $\iota=\{\,(P,x\,;x\,,P):P\in\Omega\text{ and }x\in T\,\}$ and $\epsilon=\operatorname{ev}|\,(T\times\bigcup\mathcal{F})$, we have $f=\epsilon\circ(\operatorname{id}T\underset{2}{\times}f^{\vee})\circ\iota$ with continuous maps

$$\iota:\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T}\to\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U},\text{ hence }\mathcal{W}=\mathcal{U}\underset{\mathrm{k}}{\times}\mathcal{T}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U},\qquad\text{and}$$

$$(\operatorname{id}T\underset{2}{\times}f^{\vee}):\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U}\to\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{F},\text{ hence }\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{F},\qquad\text{and}$$

$\epsilon:\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{F}\to\mathcal{T}_1$ by Proposition 22 above. The assertion follows. □

24 Corollary. *Let* $(\Omega,\mathcal{U}),(T,\mathcal{T}),(T_1,\mathcal{T}_1)$ *be any topological spaces such that* $\mathcal{U}$ *is almost compactly generated and* $\mathcal{T}$ *is locally compact, and let* $f\in{T_1}^{\Omega\times T}$ *and* $\mathcal{P}=(\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T},\mathcal{T}_1)$. *If* $(\mathcal{P},f\,|\,(K\times\mathbf{U}))$ *is a topological map for all* $\mathcal{U}$*–compact* K, *then also* $(\mathcal{P},f)$ *is a topological map.*

Proof. Assuming the premise and putting $\mathcal{F}=\mathrm{t}C_{\mathrm{co}}(\mathcal{T},\mathcal{T}_1)$, by Proposition 23(a) it suffices that $(\mathcal{U},\mathcal{F},f^{\vee})$ is a topological map, which in turn follows if $(\mathcal{U}\cap K,\mathcal{F},f^{\vee}|\,K)$ is a such for all $\mathcal{U}-$compact K. Again by Proposition 23(a) it suffices that $(f^{\vee}|\,K)^{\wedge}=f\,|\,(K\times\mathbf{U})$ is continuous $(\mathcal{U}\cap K)\underset{\mathrm{t}}{\times}\mathcal{T}\to\mathcal{T}_1$, but this is immediate from the premise. □

25 Corollary. *If* $\mathcal{T}$ *and* $\mathcal{U}$ *are any topologies of which one is compactly generated and the other is locally compact, then* $\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}=\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U}$ *holds.*

Proof. Letting $P=\bigcup\mathcal{T}\times\bigcup\mathcal{U}$ and $\iota_0=\{\,(x\,,y\,;y\,,x):(x\,,y)\in P\,\}$, we have $(\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U},\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T},\iota_0)$ a homeomorphism, hence also $(\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U},\mathcal{U}\underset{\mathrm{k}}{\times}\mathcal{T},\iota_0)$ by Lemma 15 above. Consequently, if we prove the assertion for $\mathcal{U}$ compactly generated and $\mathcal{T}$ locally compact, it also holds with the roles reversed. Now, letting the roles be as suggested, it suffices that $\operatorname{id}P$ is continuous $\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}$, equivalently that ${\iota_0}^{-\iota}$ is continuous $\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}$, and for this by Corollary 24 it suffices that for every $\mathcal{U}-$compact K we have ${\iota_0}^{-\iota}|\,(K\times\mathbf{U})$ continuous $\mathcal{U}\underset{\mathrm{t}}{\times}\mathcal{T}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}$, equivalently $\operatorname{id}(\bigcup\mathcal{T}\times K)$ continuous $\mathcal{T}\underset{\mathrm{t}}{\times}(\mathcal{U}\cap K)\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}$. For this, since by [13; Theorem 5.17, p. 146] the topology $\mathcal{U}\cap K$ is locally compact in the sense of our Definitions 12 above, further by Corollary 24 it suffices that for every $\mathcal{T}-$compact B we have $\operatorname{id}(B\times K)$ continuous $\mathcal{T}\underset{\mathrm{t}}{\times}\mathcal{U}\to\mathcal{T}\underset{\mathrm{k}}{\times}\mathcal{U}$, but this is trivial. □

Our Corollary 25 above may be compared to the slightly weaker result in [4; Theorem 4.4, p. 263] with less formal and six lines longer presentation of proof where both topologies are required to be Hausdorff.

26 Proposition. *It holds that* $\kappa_{\mathrm{tv}}``\mathrm{LCS}(\boldsymbol{R})\subseteq\mathrm{cgVS}(\boldsymbol{R})$. *For every sequentially complete* $E\in\mathrm{LCS}(\boldsymbol{R})$ *it holds that* $\kappa_{\mathrm{tv}}`E\in\mathrm{scgVS}(\boldsymbol{R})$.

Proof. Arbitrarily given $E=(a\,,c\,,\mathcal{T})\in\mathrm{LCS}(\boldsymbol{R})$, for $\mathcal{U}=k_{\mathrm{elt}}\mathcal{T}$ we have to prove that a is continuous $\mathcal{U}\underset{\mathrm{k}}{\times}\mathcal{U}\to\mathcal{U}$ and c is continuous $\tau_{\mathbb{R}}\underset{\mathrm{t}}{\times}\mathcal{U}\to\mathcal{U}$. Using Lemmas 15 and 16 and Corollary 25 above, one easily deduces these. □

In particular, from the preceding proposition we see that the Seip convenient spaces are exactly the $\kappa_{\mathrm{tv}}\grave{}E$ for some sequentially complete real Hausdorff locally convex space E. Again using Lemma 16 and noting that $\tau_{I\!R}$ and $\tau_{I\!R} \underset{\mathrm{t}}{\times} \tau_{I\!R}$ are compactly generated, and that sequential completeness is preserved when taking products of small families of locally convex spaces, we get the following

27 Corollary. *Let $\mathcal{O} = \mathrm{scgVS}(\boldsymbol{R})$. Then $\boldsymbol{R} \in \mathcal{O}$, and for any $\boldsymbol{E}, E, F, I$ the implications $E, F \in \mathcal{O} \Rightarrow E \boxtimes_{\mathrm{k}} F \in \mathcal{O}$ and $\boldsymbol{E} \in \mathcal{O}^{\,I} \Rightarrow \prod_{\mathrm{cgVS}\,I\!R} \boldsymbol{E} \in \mathcal{O}$ hold.*

We shall need the spaces of continuous linear maps given in the following

28 *Constructions.*

(1) $\mathcal{L}(E, F) = \{\, \ell : \ell$ linear $\sigma_{\mathrm{rd}} E \to \sigma_{\mathrm{rd}} F$ and $\ell^{-\iota}\grave{}\grave{}\grave{}\,\tau_{\mathrm{rd}} F \subseteq \tau_{\mathrm{rd}} E \,\}$,

(2) $\mathcal{L}_{\mathrm{co}}(E, F) = \bigcap \{\, \mathrm{vF}_{\mathrm{so}}\, \mathcal{L}(E, F)_{\mathcal{G}}\, F : E : \mathcal{G} = \{\, K : K$ is $\tau_{\mathrm{rd}} E-$compact $\}\}$,

(3) $\mathcal{L}_{k}(E, F) = \bigcap \{\, \mathrm{vF}_{\mathrm{su}}\, \mathcal{L}(E, F)_{\mathcal{G}}\, F : E : \mathcal{G} = \{\, K : K$ is $\tau_{\mathrm{rd}} E-$compact $\}\}$,

(4) $\mathcal{L}_{\mathrm{cg}}(E, F) = \bigcap \{\, \kappa_{\mathrm{tv}}\grave{}\,\mathrm{vF}_{\mathrm{so}}\, C_{\mathcal{G}}\, F_{/\,S} : F : C = \bigcup \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, \tau_{\mathrm{rd}} F)$ and $\mathcal{G} = \{\, K : K$ is $\tau_{\mathrm{rd}} E-$compact $\}$ and $S = \mathcal{L}(E, F)\,\}$.

29 Lemma. *Let $\mathcal{T} = \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, \tau_{\mathrm{rd}} F)$ where $F \in \mathrm{cgVS}(\boldsymbol{R})$ and E is any real topologized vector space. Then $\mathcal{L}(E, F)$ is $k_{\mathrm{elt}}\mathcal{T}-$closed.*

Proof. For $\mathrm{E}\, x\, y\, t = \langle\, (u\grave{}(x+y) - u\grave{}x - u\grave{}y, u\grave{}(t\,x) - t\,(u\grave{}x)) : u \in \bigcup \mathcal{T} \,\rangle$, we have $\mathcal{L}(E, F) = \bigcap \{\, (\mathrm{E}\, x\, y\, t)^{-\iota}\grave{}\grave{}\{\,(\mathbf{0}_F, \mathbf{0}_F)\}: x, y \in v_s E$ and $t \in I\!R\,\}$. For fixed $z \in v_s E$, directly by definition, we see that $\mathrm{ev}_z \,|\, \bigcup \mathcal{T}$ is continuous $\mathcal{T} \to \tau_{\mathrm{rd}} F$. For $x, y \in v_s E$ and $t \in I\!R$, letting $f = \langle\, (u\grave{}(x+y), u\grave{}x, u\grave{}y, u\grave{}(t\,x)) : u \in \bigcup \mathcal{T} \,\rangle$ and $\mathcal{U} = \tau_{\mathrm{rd}} F \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F$, it follows that f is continuous $\mathcal{T} \to \mathcal{U}$, and by Lemma 15 hence $k_{\mathrm{elt}}\mathcal{T} \to k_{\mathrm{elt}}\mathcal{U}$. Taking into account that $F \in \mathrm{cgVS}(\boldsymbol{R})$, and using Lemma 16 one deduces that $\mathrm{E}\, x\, y\, t$ is continuous $k_{\mathrm{elt}}\mathcal{T} \to \tau_{\mathrm{rd}} F \underset{\mathrm{t}}{\times} \tau_{\mathrm{rd}} F$. The assertion of the lemma now follows. □

30 Proposition. *Let $E, F \in \mathrm{cgVS}(\boldsymbol{R})$ and $F_1 \in \mathrm{LCS}(\boldsymbol{R})$. Then*

(1) $\mathcal{L}_{\mathrm{cg}}(E, F) = \kappa_{\mathrm{tv}}\grave{}\mathcal{L}_{\mathrm{co}}(E, F) \in \mathrm{cgVS}(\boldsymbol{R})$,

(2) $\mathcal{L}_{k}(E, F_1) = \mathcal{L}_{\mathrm{co}}(E, F_1) \in \mathrm{LCS}(\boldsymbol{R})$,

(3) $\mathcal{L}_{\mathrm{cg}}(E, \kappa_{\mathrm{tv}}\grave{}F_1) = \kappa_{\mathrm{tv}}\grave{}\mathcal{L}_{k}(E, F_1) \in \mathrm{cgVS}(\boldsymbol{R})$,

(4) $F \in \mathrm{scgVS}(\boldsymbol{R}) \Rightarrow \mathcal{L}_{\mathrm{cg}}(E, F) \in \mathrm{scgVS}(\boldsymbol{R})$.

Proof. For (1) letting $L = \mathcal{L}_{\mathrm{cg}}(E, F)$ and $\mathcal{U} = \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, \tau_{\mathrm{rd}} F)$, by an elementary set theoretic verification the proof of the asserted equality is reduced to getting $k_{\mathrm{elt}}\mathcal{U} \cap v_s L = k_{\mathrm{elt}}(\mathcal{U} \cap v_s L)$. For $k_{\mathrm{elt}}(\mathcal{U} \cap v_s L) \subseteq k_{\mathrm{elt}}\mathcal{U} \cap v_s L$, we note that trivially $\mathrm{id}_{\mathrm{v}} L$ is continuous $k_{\mathrm{elt}}\mathcal{U} \cap v_s L \to \mathcal{U} \cap v_s L$. By Lemmas 29 and 17 and 15 it is also continuous $k_{\mathrm{elt}}\mathcal{U} \cap v_s L \to k_{\mathrm{elt}}(\mathcal{U} \cap v_s L)$, whence the inclusion. For the converse using Lemma 15 note the continuity implications $\mathcal{U} \cap v_s L \to \mathcal{U}$, hence $k_{\mathrm{elt}}(\mathcal{U} \cap v_s L) \to k_{\mathrm{elt}}\mathcal{U}$, hence $k_{\mathrm{elt}}(\mathcal{U} \cap v_s L) \to k_{\mathrm{elt}}\mathcal{U} \cap v_s L$.

To get $L \in \mathrm{cgVS}(\boldsymbol{R})$ in (1), also let $G = \kappa_{\mathrm{tv}}\grave{}\,\mathrm{vF}_{\mathrm{so}} \bigcup \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, \tau_{\mathrm{rd}} F)_{\mathcal{K}}\, F$ where $\mathcal{K} = \{\, K : K$ is $\tau_{\mathrm{rd}} E-$compact $\}$. If $G \in \mathrm{cgVS}(\boldsymbol{R})$ holds, since $L = G_{/\,v_s L}$, we immediately get $L \in \mathrm{cgVS}(\boldsymbol{R})$ by Lemma 29 and Corollary 18 above. It hence suffices to establish $G \in \mathrm{cgVS}(\boldsymbol{R})$ which in turn follows if $(\mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{T}, \mathcal{T}, \sigma_{\mathrm{rd}}^2 G)$ and $(\tau_{I\!R} \underset{\mathrm{t}}{\times} \mathcal{T}, \mathcal{T}, \tau\sigma_{\mathrm{rd}} G)$ are topological maps when $\mathcal{T} = \tau_{\mathrm{rd}} G = k_{\mathrm{elt}}\mathcal{U}$.

To prove continuity of $\sigma_{\mathrm{rd}}^2 G : \mathcal{T} \underset{\mathrm{k}}{\times} \mathcal{T} \to \mathcal{T}$, by Proposition 23 and Lemma 15 it suffices that $\sigma_{\mathrm{rd}}^2 G^{\wedge}$ is continuous $\tau_{\mathrm{rd}}(G \boxtimes_{\mathrm{k}} G \boxtimes_{\mathrm{k}} E) \to \tau_{\mathrm{rd}} F$. For $\epsilon = \mathrm{ev} \,|\, v_s(E \sqcap G)$ and $\iota_2 = \{\, (u, v, x, (x, u\,; x, v)) : u, v \in v_s G$ and $x \in v_s E \,\}$, observing that we have $\sigma_{\mathrm{rd}}^2 G^{\wedge} = \sigma_{\mathrm{rd}}^2 F \circ (\epsilon \underset{2}{\times} \epsilon) \circ \iota_2$, since ϵ is continuous $\tau_{\mathrm{rd}}(E \boxtimes_{\mathrm{k}} G) \to \tau_{\mathrm{rd}} F$ by

Proposition 22 above, it suffices to note that $(G \boxed{\scriptstyle k} G \boxed{\scriptstyle k} E\,, E \boxed{\scriptstyle k} G \boxed{\scriptstyle k} (E \boxed{\scriptstyle k} G)\,, \iota_2)$ and $(F \boxed{\scriptstyle k} F, F, \sigma_{\mathrm{rd}}^2 F)$ are continuous linear maps.

For continuity of $\tau\sigma_{\mathrm{rd}}\, G : \tau_{\mathbb{R}} \underset{\mathrm{t}}{\times} \mathcal{T} \to \mathcal{T}$, as above it suffices that $\tau\sigma_{\mathrm{rd}}\, G^{\wedge}$ is continuous $\tau_{\mathrm{rd}}(\boldsymbol{R} \boxed{\scriptstyle k} G \boxed{\scriptstyle k} E) \to \tau_{\mathrm{rd}} F$, now noting that $\tau_{\mathbb{R}} \underset{\mathrm{t}}{\times} \mathcal{T}$ is compactly generated by Corollary 25 above. Since by a twofold application of Lemma 15 with $\iota_1 = \{\,(t, u, x, (t\,; x, u)) : t \in \mathbb{R}$ and $u \in \upsilon_s G$ and $x \in \upsilon_s E\,\}$ we have ι_1 continuous $\tau_{\mathrm{rd}}(\boldsymbol{R} \boxed{\scriptstyle k} G \boxed{\scriptstyle k} E) \to \tau_{\mathrm{rd}}(\boldsymbol{R} \boxed{\scriptstyle k} (E \boxed{\scriptstyle k} G))$ with $\tau\sigma_{\mathrm{rd}}\, G^{\wedge} = \tau\sigma_{\mathrm{rd}}\, F \circ (\mathrm{id}\, \mathbb{R} \underset{2}{\times} \epsilon) \circ \iota_1$, continuity of $(\tau_{\mathbb{R}} \underset{\mathrm{t}}{\times} \mathcal{T}, \mathcal{T}, \tau\sigma_{\mathrm{rd}}\, G)$ then follows.

We get (2) by applying elementary set theoretic manipulations to Constructions 19 and 28 above, and using Propositions 20 and 21 with F_1 in place of F and also taking $\Omega = \upsilon_s E$ and $S = \mathcal{L}(E, F_1)$.

For (3) letting $L_1 = \mathcal{L}_{\mathrm{cg}}(E, \kappa_{\mathrm{tv}}{}^\backprime F_1)$ and $L_0 = \mathcal{L}_{\mathrm{co}}(E, F_1)$ and $L = \kappa_{\mathrm{tv}}{}^\backprime L_0$, and noting (2) we should have $L_1 = L$ since we get the membership $L_1 \in \mathrm{cgVS}(\boldsymbol{R})$ from (1) by Proposition 26 above. Since by Lemma 15 we have $\upsilon_s L_0 \subseteq \upsilon_s L_1$, we trivially get $\sigma_{\mathrm{rd}} L = \sigma_{\mathrm{rd}} L_0 = \sigma_{\mathrm{rd}} L_1$, and hence it remains to establish $\tau_{\mathrm{rd}} L_1 = \tau_{\mathrm{rd}} L$. To get the simpler $\tau_{\mathrm{rd}} L \subseteq \tau_{\mathrm{rd}} L_1$, we should have (u) that $\mathrm{id}_{\mathrm{v}} L$ is continuous $\tau_{\mathrm{rd}} L_1 \to \tau_{\mathrm{rd}} L$. Now, for $\mathcal{T}_0 = \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, \tau_{\mathrm{rd}} F_1)$ and $\mathcal{T}_1 = \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} E, k_{\mathrm{elt}}\tau_{\mathrm{rd}} F_1)$ we trivially have $\mathrm{id} \bigcup \mathcal{T}_1$ continuous $k_{\mathrm{elt}}\mathcal{T}_1 \to \mathcal{T}_0$, and hence $\mathrm{id}_{\mathrm{v}} L$ continuous $\tau_{\mathrm{rd}} L_1 = k_{\mathrm{elt}}\mathcal{T}_1 \cap \upsilon_s L \to \mathcal{T}_0$, whence also $\tau_{\mathrm{rd}} L_1 \to \mathcal{T}_0 \cap \upsilon_s L = \tau_{\mathrm{rd}} L_0$. By Lemma 15 the required (u) immediately follows.

For $\tau_{\mathrm{rd}} L_1 \subseteq \tau_{\mathrm{rd}} L$ in (3) we show that $\mathrm{id}_{\mathrm{v}} L$ is continuous $\tau_{\mathrm{rd}} L \to \tau_{\mathrm{rd}} L_1$. Having $\tau_{\mathrm{rd}} L_1 = k_{\mathrm{elt}}\mathcal{T}_1 \cap \upsilon_s L$ and $\tau_{\mathrm{rd}} L$ compactly generated (again by Lemma 15 above) it suffices to establish continuity $\tau_{\mathrm{rd}} L \to \mathcal{T}_1$. By Proposition 23 (b) above, this follows if we show that $\mathrm{id}_{\mathrm{v}} L^{\wedge}$ is continuous $\tau_{\mathrm{rd}}(L \boxed{\scriptstyle k} E) \to k_{\mathrm{elt}}\tau_{\mathrm{rd}} F_1$ which in turn follows from continuity $\tau_{\mathrm{rd}}(L \boxed{\scriptstyle k} E) \to \tau_{\mathrm{rd}} F_1$. Noting that $\mathrm{id}_{\mathrm{v}} L^{\wedge}$ is defined by $(\ell, x) \mapsto \ell{}^\backprime x$, and putting $P = \upsilon_s(E \sqcap L)$ and $Q = \upsilon_s E \times \bigcup \mathcal{T}_0$ and $\epsilon = \mathrm{ev}\,|\,Q$, it suffices (e) that $\epsilon\,|\,P$ is continuous $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} L) \to \tau_{\mathrm{rd}} F_1$. Now, from Proposition 22 (b) we know that ϵ is continuous $\tau_{\mathrm{rd}} E \underset{\mathrm{k}}{\times} \mathcal{T}_0 \to \tau_{\mathrm{rd}} F_1$, and hence that $\epsilon\,|\,P$ is continuous $\tau_{\mathrm{rd}} E \underset{\mathrm{k}}{\times} \mathcal{T}_0 \cap P \to \tau_{\mathrm{rd}} F_1$. Consequently (e) follows if $\mathrm{id}\, P$ is continuous $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} L) \to \tau_{\mathrm{rd}} E \underset{\mathrm{k}}{\times} \mathcal{T}_0 \cap P$, following from continuity $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} L) \to \tau_{\mathrm{rd}} E \underset{\mathrm{t}}{\times} \mathcal{T}_0$, which indeed is the case by

$$\tau_{\mathrm{rd}} E \underset{\mathrm{t}}{\times} (\mathcal{T}_0 \cap \upsilon_s L) = \tau_{\mathrm{rd}} E \underset{\mathrm{t}}{\times} (\tau_{\mathrm{rd}} L_0) \subseteq$$
$$\tau_{\mathrm{rd}} E \underset{\mathrm{t}}{\times} (k_{\mathrm{elt}}\tau_{\mathrm{rd}} L_0) = \tau_{\mathrm{rd}} E \underset{\mathrm{t}}{\times} (\tau_{\mathrm{rd}} L) \subseteq \tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} L)\,.$$

For (4) assuming $F \in \mathrm{scgVS}(\boldsymbol{R})$, a sequentially complete $F_1 \in \mathrm{LCS}(\boldsymbol{R})$ exists with $F = \kappa_{\mathrm{tv}}{}^\backprime F_1$ whence by (3) we have $\mathcal{L}_{\mathrm{cg}}(E, F) = \kappa_{\mathrm{tv}}{}^\backprime \mathcal{L}_k(E, F_1)$. By a suitable adaptation of the proof in [13; Theorem 7.12, p. 231] or [16; Satz 1.43, p. 22], the locally convex space $\mathcal{L}_k(E, F_1)$ is sequentially complete. Hence the assertion follows from Proposition 26 above. □

31 Proposition. *Let $E, F, G \in \mathrm{cgVS}(\boldsymbol{R})$ and $U \in \tau_{\mathrm{rd}} E$ and $f_1 \in \big((\upsilon_s G)^{\,\upsilon_s F}\big)^U$ be such that $f_1{}^\backprime x$ is linear $\sigma_{\mathrm{rd}} F \to \sigma_{\mathrm{rd}} G$ for all $x \in U$. Then $f_1{}^{\wedge}$ is continuous $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} F) \to \tau_{\mathrm{rd}} G$ if and only if f_1 is continuous $\tau_{\mathrm{rd}} E \to \tau_{\mathrm{rd}} \mathcal{L}_{\mathrm{cg}}(F, G)$.*

Proof. Let $L = \mathcal{L}_{\mathrm{cg}}(F, G)$ and $\mathcal{T} = \mathrm{t}C_{\mathrm{co}}(\tau_{\mathrm{rd}} F, \tau_{\mathrm{rd}} G)$ and $W = U \times \upsilon_s F$. By Lemma 17 then $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} F) \cap W$ and $\tau_{\mathrm{rd}} E \cap U$ are compactly generated. In view of Lemma 15 it follows that $\tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} F) \cap W = \tau_{\mathrm{rd}} E \cap U \underset{\mathrm{k}}{\times} \tau_{\mathrm{rd}} F$. Using also Proposition 23 (b) in $\overset{!}{\Leftrightarrow}$ below, we get the equivalences

$$\begin{aligned} & f_1{}^{\wedge} \text{ is continuous } \tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} F) \to \tau_{\mathrm{rd}} G \\ \Leftrightarrow\ & f_1{}^{\wedge} \text{ is continuous } \tau_{\mathrm{rd}}(E \boxed{\scriptstyle k} F) \cap W \to \tau_{\mathrm{rd}} G \\ \Leftrightarrow\ & f_1{}^{\wedge} \text{ is continuous } \tau_{\mathrm{rd}} E \cap U \underset{\mathrm{k}}{\times} \tau_{\mathrm{rd}} F \to \tau_{\mathrm{rd}} G \end{aligned}$$

$$\overset{!}{\Leftrightarrow}\ f_1 \ \text{ is continuous } \tau_{\rm rd}\, E \cap U \to \mathcal{T}$$
$$\Leftrightarrow\ f_1 \ \text{ is continuous } \tau_{\rm rd}\, E \cap U \to k_{\rm elt}\mathcal{T}$$
$$\Leftrightarrow\ f_1 \ \text{ is continuous } \tau_{\rm rd}\, E \to k_{\rm elt}\mathcal{T} \cap \upsilon_s L = \tau_{\rm rd}\, L\,. \qquad \square$$

32 *Definition.* The class of continuous bilinear maps of Seip−convenient spaces is

$$\begin{aligned} {}^{\rm bi}\mathcal{L}_{\rm scgVS}(\boldsymbol{R}) = \{\, & (E \boxtimes_k F, G, b) : E, F, G \in {\rm scgVS}(\boldsymbol{R}) \text{ and} \\ & b \text{ function } \upsilon_s(E \sqcap F) \to \upsilon_s G \text{ and } b \text{ continuous } \tau_{\rm rd}(E \boxtimes_k F) \to \tau_{\rm rd}\, G \\ & \text{and } \forall\, x, y\,;\ [\ x \in \upsilon_s E \Rightarrow b\,(x,\cdot) \text{ linear } \sigma_{\rm rd}\, F \to \sigma_{\rm rd}\, G\ ] \\ & \text{and } [\ y \in \upsilon_s F \Rightarrow b\,(\cdot, y) \text{ linear } \sigma_{\rm rd}\, E \to \sigma_{\rm rd}\, G\ ]\,\}\,. \end{aligned}$$

33 Proposition. *Let $E, F \in {\rm cgVS}(\boldsymbol{R})$ and $G = E \boxtimes_k \mathcal{L}_{\rm cg}(E, F)$ and $\epsilon = {\rm ev} \,|\, \upsilon_s G$. Then ϵ is continuous $\tau_{\rm rd}\, G \to \tau_{\rm rd}\, F$. If in addition $E, F \in {\rm scgVS}(\boldsymbol{R})$, the membership $(G, F, \epsilon) \in {}^{\rm bi}\mathcal{L}_{\rm scgVS}(\boldsymbol{R})$ also holds.*

Proof. Let $L = \mathcal{L}_{\rm co}(E, F)$ and $\mathcal{T} = {\rm t}C_{\rm co}(\tau_{\rm rd}\, E, \tau_{\rm rd}\, F)$. By Proposition 22(b) we have ${\rm ev} \,|\, (\upsilon_s E \times \bigcup \mathcal{T})$ continuous $\tau_{\rm rd}\, E \underset{\rm k}{\times} \mathcal{T} \to \tau_{\rm rd}\, F$, and hence ϵ continuous $\tau_{\rm rd}\, E \underset{\rm k}{\times} \mathcal{T} \cap \upsilon_s G \to \tau_{\rm rd}\, F$. For the first assertion it hence suffices (c) that ${\rm id}_{\rm v} G$ is continuous $\tau_{\rm rd}\, G \to \tau_{\rm rd}\, E \underset{\rm k}{\times} \mathcal{T} \cap \upsilon_s G$. Recalling Lemma 16 and and using (1) of Proposition 30 we see that $\tau_{\rm rd}\, G = k_{\rm elt}(\tau_{\rm rd}\, E \underset{\rm t}{\times} k_{\rm elt}\tau_{\rm rd}\, L) = k_{\rm elt}(\tau_{\rm rd}\, E \underset{\rm t}{\times} \tau_{\rm rd}\, L) = k_{\rm elt}(\tau_{\rm rd}\, E \underset{\rm t}{\times} (\mathcal{T} \cap \upsilon_s L))$, whence by Lemma 15 then (c) follows. For the latter assertion just note also (4) of Proposition 30 above. $\square$

34 Lemma. *Let $F \in {\rm cgVS}(\boldsymbol{R})$ and $G \in {\rm LCS}(\boldsymbol{R})$. If $\boldsymbol{\ell} \to \ell$ in top $\tau_{\rm rd}\, \mathcal{L}_k(F, G)$, then also $\boldsymbol{\ell} \to \ell$ in top $\tau_{\rm rd}\, \mathcal{L}_{\rm cg}(F, \kappa_{\rm tv}\grave{}\, G)$.* (sequential convergences!)

Proof. Let $H = \kappa_{\rm tv}\grave{}\, G$ and $L = \mathcal{L}_{\rm co}(F, H)$ and $\mathcal{T} = \tau_{\rm rd}\, \mathcal{L}_{\rm co}(F, G)$. By Proposition 26 and (1) and (2) of Proposition 30 then $\mathcal{T} = \tau_{\rm rd}\, \mathcal{L}_k(F, G)$ and $\kappa_{\rm tv}\grave{}\, L = \mathcal{L}_{\rm cg}(F, H)$. Since the topologies $\tau_{\rm rd}\, L$ and $k_{\rm elt}\tau_{\rm rd}\, L$ have the same compact sets, also the *sequential* convergence $\boldsymbol{\ell} \to \ell$ means the same in both topologies. Hence, assuming (c) that $\boldsymbol{\ell} \to \ell$ in top $\mathcal{T}$, it suffices to prove that $\boldsymbol{\ell} \to \ell$ in top $\tau_{\rm rd}\, L$. For this, arbitrarily given a $\tau_{\rm rd}\, F$−compact K and V with $\ell\,``K \subseteq V \in \tau_{\rm rd}\, H$, we should establish (e) existence of $N \in I\!N_{\rm o}$ such that for ${\rm C}\, N = \bigcup(\boldsymbol{\ell}\,``(I\!N_{\rm o} \setminus N))``K$ we have ${\rm C}\, N \subseteq V$. To get this, putting $B = {\rm rng}\, \boldsymbol{\ell} \cup \{\ell\}$ and $A = \bigcup B``K$, then B is $\mathcal{T}$−compact. Considering $\epsilon = {\rm ev} \,|\, (K \times \bigcup \mathcal{T})$, like in Proposition 22(a) one verifies that ϵ is continuous $\tau_{\rm rd}\, F \underset{\rm t}{\times} \mathcal{T} \to \tau_{\rm rd}\, G$. Also noting that $A = \epsilon\,[\,K \times B\,]$, it follows that A is $\tau_{\rm rd}\, G$−compact. Consequently, there is $U \in \tau_{\rm rd}\, G$ such that $U \cap A = V \cap A$. As also $\ell\,``K \subseteq A$ holds, we have $\ell\,``K \subseteq V \cap A = U \cap A \subseteq U$, and hence by (c) there is $N \in I\!N_{\rm o}$ with ${\rm C}\, N \subseteq U$. Also having ${\rm C}\, N \subseteq A$, we get ${\rm C}\, N \subseteq U \cap A = V \cap A \subseteq V$, as required in (e) above. $\square$

C. Riemann integration of curves in topologized vector spaces

In some proofs below, we shall need to integrate a (continuous) curve on the unit interval $[\,0,1\,]$ taking values in a compactly generated or locally convex space. To be able to treat these in a unified manner accordant with our general approach in this paper, we here give a short account on this matter.

35 *Definition.* Let I be a nontrivial real compact interval, i.e. for some $a, b \in I\!R$ with $a < b$, we have $I = [\,a, b\,] = \{\,t : a \le t \le b\,\}$. Let $E = (X, \mathcal{T})$ be a real topologized vector space. We then say that γ on I in E is *Riemann integrable* to x if and only if γ is a function with $I \subseteq {\rm dom}\, \gamma$ and ${\rm rng}\, \gamma \subseteq \upsilon_s E$, and $x \in \upsilon_s E$ is such

that for every $V \in \mathcal{N}_{bh}(x, \mathcal{T})$ there is $\delta \in I\!R^+$ with the property that for all $k \in I\!N$ and for all $\boldsymbol{t} \in I^{k+1\cdot}$ and $\boldsymbol{s} \in I^k$ satisfying

$$I = [\,\boldsymbol{t}`\emptyset, \boldsymbol{t}`k\,] \quad \text{and} \quad k \subseteq \{\, i : \boldsymbol{t}`i \le \boldsymbol{s}`i \le \boldsymbol{t}`(i^+) \text{ and } \boldsymbol{t}`(i^+) - (\boldsymbol{t}`i) < \delta \,\},$$

we have $\sigma_{\mathrm{rd}} E\text{-}\sum_{i \in k}(\boldsymbol{t}`(i^+) - (\boldsymbol{t}`i))\,(\gamma \circ \boldsymbol{s}`i) \in V$. We also define

$$E\text{-}\int_a^b \gamma = \bigcap\{\, x : \forall z\,;\, x = z \Leftrightarrow \gamma \text{ on } [\,a, b\,] \text{ in } E \text{ is Riemann integrable to } z \,\},$$

and $E\text{-}\int_a^b \mathfrak{F}\,\mathrm{d}\,\mathfrak{s} = E\text{-}\int_a^b \langle\, \mathfrak{F} : \mathfrak{s} \in [\,a, b\,]\,\rangle_{\mathrm{old}}$ whenever $\mathfrak{F}$ is any term, and $\mathfrak{s}$ is any variable. In a connection where "E" may be considered as being implicitly understood, we may drop "E-" from the notation.

We do not formulate as an explicit proposition the obvious fact that if γ on I in F is Riemann integrable to x and to y, then $x = y$ in the case where $\tau_{\mathrm{rd}} F$ is a Hausdorff topology. Even without any Hausdorff assumption, either we have that γ on $[\,a, b\,]$ in F is Riemann integrable to $F\text{-}\int_a^b \gamma$, or else $F\text{-}\int_a^b \gamma = \mathbf{U}$. The latter in particular holds if γ is integrable to no x or to several such.

36 Lemma. *Let $(F_1, F) \in \kappa_{\mathrm{tv}} | \mathrm{LCS}(\boldsymbol{R})$. Then γ on I in F is Riemann integrable to x if and only if the same holds in the space F_1.*

Proof. By $\tau_{\mathrm{rd}} F_1 \subseteq \tau_{\mathrm{rd}} F$, integrability in F trivially implying that in F_1, it suffices to show that a contradiction follows if γ on I in F_1 is Riemann integrable to x but not in F. Indeed, then there is $V \in \mathcal{N}_{bh}(x, \tau_{\mathrm{rd}} F)$ such that for every $i \in I\!N_{\mathrm{o}}$, considering $\delta = (i^{\bullet} + 1)^{-1}$, there are $\boldsymbol{t}, \boldsymbol{s}$ with the properties in Definition 35 such that for the Riemann sum $x_1 = \sigma_{\mathrm{rd}} F\text{-}\sum_{l \in k}(\boldsymbol{t}`(l^+) - (\boldsymbol{t}`l))\,(\gamma \circ \boldsymbol{s}`l)$ we have $x_1 \notin V$. By the axiom of choice we obtain $\boldsymbol{x} \in (\upsilon_s F_1 \setminus V)^{I\!N_{\mathrm{o}}}$ with $\boldsymbol{x}`i$ corresponding to the x_1 in the preceding for every $i \in I\!N_{\mathrm{o}}$. By integrability in F_1, we have $\boldsymbol{x} \to x$ in top $\tau_{\mathrm{rd}} F_1$. Putting $K = \mathrm{rng}\,\boldsymbol{x} \cup \{x\}$, it holds that K is $\tau_{\mathrm{rd}} F_1-$compact, hence also $\tau_{\mathrm{rd}} F-$compact. From $\tau_{\mathrm{rd}} F \cap K = \tau_{\mathrm{rd}} F_1 \cap K$, it follows that for some $V_1 \in \tau_{\mathrm{rd}} F_1$ we have $x \in V_1 \cap K \subseteq V \cap K$, and consequently for large i it holds that $\boldsymbol{x}`i \in V_1 \cap K \subseteq V \cap K \subseteq V$ in contradiction with $\boldsymbol{x}`i \notin V$ for all i. □

37 Proposition. *Let $F \in \mathrm{LCS}(\boldsymbol{R})$ be sequentially complete. Let I be a notrivial real compact interval and let $\gamma \in (\upsilon_s F)^I$ be continuous $\tau_{I\!R} \to \tau_{\mathrm{rd}} F$. Then there is x such that γ on I in F is Riemann integrable to x.*

Proof. Letting $I = [\,a, b\,]$, and considering $\boldsymbol{x} \in (\upsilon_s F)^{I\!N_{\mathrm{o}}}$ defined by

$$i \mapsto \sigma_{\mathrm{rd}} F\text{-}\sum_{l \in i^+}(i^{\bullet} + 1)^{-1}(b - a)\,(\gamma`(a + l\,(i^{\bullet} + 1)^{-1}(b - a))),$$

using continuity of γ and compactness of I and local convexity of F, one first verifies that $\boldsymbol{x}$ is a Cauchy sequence in F, whence by the assumed sequential completeness there is x with $\boldsymbol{x} \to x$ in top $\tau_{\mathrm{rd}} F$. By the same token, one verifies that γ integrates to x, the details again being left to the reader. □

38 Lemma. *Let U be closed and convex in E with $E \in \mathrm{LCS}(\boldsymbol{R})$. If also γ on $[\,0, 1\,]$ in E is Riemann integrable to y with $\mathrm{rng}\,\gamma \subseteq U$, then $y \in U$.*

Proof. Put $I = [\,0, 1\,]$, and first note the following. If $\varphi \in]\,0, +\infty\,[^{\,I}$ is integrable in the sense of Lebesgue, then $\int_I \varphi\,\mathrm{d}\,\mu_{\mathrm{Leb}} > 0$. Riemann integrability implying that of Lebesgue, if $r \in I\!R$ and $\chi \in]\,r, +\infty\,[^{\,I}$ is Riemann integrable, considering $\varphi = \langle\, \chi`t - r : t \in I\,\rangle$, we get $\boldsymbol{R}\text{-}\int_0^1 \chi = \int_I \chi\,\mathrm{d}\,\mu_{\mathrm{Leb}} = r + \int_I \varphi\,\mathrm{d}\,\mu_{\mathrm{Leb}} > r$. Now, under the premise of the lemma, if we have $y \notin U$, by Hahn−Banach, there are $r \in I\!R$ and $\ell \in \mathcal{L}(E, \boldsymbol{R})$ with $\ell``U \subseteq]\,r, +\infty\,[$ and $\ell`y = r$, and the preceding applied to $\chi = \ell \circ \gamma$ gives $r = \ell`y = \boldsymbol{R}\text{-}\int_0^1 \chi > r$, a contradiction. □

Note that the preceding Hahn – Banach argument of the proof fails if the convex set U is required to be neither closed nor open. Call a set $C \subseteq v_s E$ *countably convex* iff for any $\boldsymbol{x} \in C^{I\!N_{\rm o}}$ and $\boldsymbol{s} \in [\,0\,,1\,]^{I\!N_{\rm o}}$ and any x with $\sum \boldsymbol{s} = 1$ and $\big\langle \sum_{l \in i^+} \boldsymbol{s}\,`\,l\,(\boldsymbol{x}\,`\,l\,) : i \in I\!N_{\rm o} \big\rangle \to x$ in top $\tau_{\rm rd}\, E$ we have $x \in C$. *Does the conclusion in* Lemma 38 *remain valid if U is required to be merely countably convex?*

For example, let $\mathrm{e}\, i = \{\,(\emptyset\,,-1)\,,(i\,,1)\,\} \cup ((I\!N \setminus \{i\}) \times \{0\})$ for $i \in I\!N$, and put $\mathrm{e}\,\emptyset = \{\,(\emptyset\,,1)\,\} \cup (I\!N \times \{0\})$, and in the Fréchet space $E = (X\,,\mathcal{T}\,) = \boldsymbol{R}^{\,I\!N_{\rm o}\,]_{\rm tvs}}$ consider the countably convex set

$$U = \{\,\{\,(\emptyset\,,2\,\boldsymbol{s}\,`\,\emptyset - \textstyle\sum \boldsymbol{s})\,\} \cup (\boldsymbol{s} \,|\, I\!N) : \boldsymbol{s} \in [\,0\,,1\,]^{I\!N_{\rm o}} \text{ and } 0 < \sum \boldsymbol{s} \le 1\,\}$$

for which we have $\mathbf{0}_E \in \mathrm{Cl}_{\mathcal{T}}\, U \setminus U$. If γ on $[\,0\,,1\,]$ in E is Riemann (or even "scalarwise" Lebesgue) integrable to x with rng $\gamma \subseteq U$, is then $x = \mathbf{0}_E$ impossible?

Note that there does not exist $\ell \in \mathcal{L}\,(E\,,\boldsymbol{R})$ with $\ell\,``\,U \subseteq I\!R^+$ since for every such ℓ there is $y \in \boldsymbol{v}_s \coprod_{\mathrm{VS}\,I\!R} (I\!N_{\rm o} \times \{\mathbf{R}\})$ with $\ell = \big\langle \sum_{i \in I\!N_{\rm o}} x\,`\,i\,(y\,`\,i) : x \in v_s E \,\big\rangle$, and hence otherwise for a sufficiently large $i \in I\!N$ we would get

$$0 < \ell\,`\,\mathrm{e}\,\emptyset = y\,`\,\emptyset = -\ell\,`\,\mathrm{e}\,i < 0\,,\ \text{a contradiction.}$$

39 *Definitions.* Let $E = (X\,,\mathcal{T}\,)$ be a real topologized vector space, and consider $c : J \to \boldsymbol{v}_s X$ with $t_0 \in J \subseteq I\!R$ such that also 0 is a $\tau_{I\!R}$ – limit point of the set $I\!R \cap \{\,t : t_0 + t \in J\,\} \setminus \{0\}$. We then say that c is *differentiable to x in E at t_0* if and only if also $x \in \boldsymbol{v}_s X$ and for every $V \in \mathcal{N}_{bh}(x\,,\mathcal{T}\,)$ there is $\delta \in I\!R^+$ with the property that for all $t \in I\!R$ with $t_0 + t \in J$ and $0 < |\,t\,| < \delta$ we have the membership $t^{-1}(c\,`(t_0+t) - c\,`\,t_0) \in V$. We let

$$\mathsf{D}_E\, c = \{\,(\,t\,,x) : \forall\, z\,;\, x = z \Leftrightarrow c \text{ is differentiable to } z \text{ in } E \text{ at } t\,\}\,,$$

when E is as above and $c \in (v_s E)^{\,\mathrm{dom}\, c}$ with $\mathrm{dom}\, c \subseteq I\!R$, otherwise putting $\mathsf{D}_E\, c = \mathbf{U}$, and say that c is *differentiable in* E iff $\mathrm{dom}\, c \subseteq \mathrm{dom}\ \mathsf{D}_E\, c \ne \mathbf{U}$.

Let $I = [\,0\,,1\,]$ and $Q = I^{\times 2\cdot}$ and $\mathcal{U} = \tau_{I\!R} \underset{\rm t}{\times} \tau_{I\!R} \cap Q$. Any $c \in (v_s E)^{\,I}$ we call a *standard differentiable curve* in E in case also $]\,0\,,1\,[\, \subseteq \mathrm{dom}\ \mathsf{D}_E\, c \ne \mathbf{U}$ holds and c is continuous $\tau_{I\!R} \to \tau_{\rm rd}\, E$ at the points 0 and 1. By a *standard family of continuously differentiable curves* in E we mean any $\Gamma \in \big((v_s E)^{\,I}\big)^{I}$ such that Γ is continuous $\tau_{I\!R} \to \tau_{\rm rd}\, E^{\,I\,]_{\rm ti}}$ and such that for $g = \langle\, \mathsf{D}_E\, (\Gamma\,`\,s)\,`\,t : \mathcal{S} = (s\,,t) \in Q\,\rangle$ we also have that g is continuous $\mathcal{U} \to \tau_{\rm rd}\, E$ with $Q \subseteq \mathrm{dom}\, g$.

40 Proposition (mean value theorem)**.** *Let U be a closed and convex set in E with $E \in \mathrm{LCS}\,(\boldsymbol{R})$. Let c be a standard differentiable curve in E with $c\,`\,0 = \mathbf{0}_E \in U$ and $\mathsf{D}_E\, c\,``\,]\,0\,,1\,[\, \subseteq U$. Then $c\,`\,1 \in U$ holds.*

Proof. To proceed by reductio ad absurdum, let $c\,`\,1 \notin U$. By Hahn – Banach, there is $\ell \in \mathcal{L}(E\,,\boldsymbol{R})$ with $\ell\,``\,U \subseteq\, ]{-\infty}\,,1\,[$ and $\ell\,`(c\,`\,1) = 1$. The classical mean value theorem when applied to the function $\ell \circ c$ gives some $t$ with $0 < t < 1$ and $1 = \ell\,`(c\,`\,1) = \mathsf{D}_{\boldsymbol{R}}\,(\ell \circ c)\,`\,t = \ell\,`(\mathsf{D}_E\, c\,`\,t) \in \ell\,``\,U \subseteq\,]{-\infty}\,,1\,[$, a contradiction. $\square$

41 Proposition. *Let $I = [\,0\,,1\,]$, and let $E \in \mathrm{LCS}\,(\boldsymbol{R})$ be sequentially complete. Let Γ be a standard family of continuously differentiable curves in E, and let $c = \big\langle\, E\text{-}\int_0^1 \Gamma\,`\,s\,`\,t \,\mathrm{d}\, s : t \in I \,\big\rangle$. Then c is a standard differentiable curve in the space E with also $\mathsf{D}_E\, c = \big\langle\, E\text{-}\int_0^1 \mathsf{D}_E\, (\Gamma\,`\,s)\,`\,t \,\mathrm{d}\, s : t \in I \,\big\rangle$.*

Proof. Let $\mathcal{U}, g$ be as in Definitions 39 above, and arbitrarily fix $t_0 \in I$, and put $x = E\text{-}\int_0^1 \mathsf{D}_E\,(\Gamma\,`\,s)\,`\,t_0 \,\mathrm{d}\, s$. Then it suffices to show that c is differentiable to x in E at t_0. Writing $\mathrm{F}\, s\, s_1\, t = t^{-1}(\Gamma\,`\,s\,`(t_0 + s_1 t) - \Gamma\,`\,s\,`\,t_0) - s_1\, \mathsf{D}_E\,(\Gamma\,`\,s)\,`\,t_0$, and arbitrarily fixing a closed convex $U \in \mathcal{N}_o\, E$, using Proposition 37 and Lemma

38 above, one quickly checks that it further suffices to show (e) existence of $\delta \in I\!\!R^+$ such that for all $t \in I\!\!R$ with $t_0 + t \in I$ and $0 < |t| < \delta$ we have $\mathrm{F}\, s\, 1\, t \in U$.

To get (e) above, we first note that using $\tau_{I\!\!R}$ – compactness of I and continuity $\mathcal{U} \to \tau_{\mathrm{rd}} E$ of g, one deduces existence of some $\delta \in I\!\!R^+$ such that for all $t \in I\!\!R$ with $t_0 + t \in I$ and $|t| < \delta$ we have

(x) $\quad \mathsf{D}_E\,(\Gamma\text{`}s)\text{`}(t_0 + t) - \mathsf{D}_E\,(\Gamma\text{`}s)\text{`}t_0 = g\text{`}(s, t_0 + t) - g\text{`}(s, t_0) \in U.$

Now arbitrarily fixing t as required in (e) we get $\mathrm{F}\, s\, 1\, t \in U$ by Proposition 40 and (x) when considering the curve $\langle\, \mathrm{F}\, s\, s_1\, t : s_1 \in I\,\rangle$. $\square$

D. Seip's higher order differentiability classes

In a rather non-uniform manner, Seip scattered his definitions of his first, higher, and infinite order continuous differentiabilities in [16; pp. 60, 75, 85]. In Definitions 42 below, we give our own reformulation of these by constructing the differentiability classes $\mathcal{D}^k_{\mathrm{Seip}}$ for $k \in \infty^+$ at a (multiple) stroke. In Theorem 55 these turn out to exactly capture Seip's concepts. Theorem 52 shows that these are obtained as particular cases of the general BGN – construction.

42 ***Definitions.*** For all classes $\tilde f, k$ first generally letting

(1) $\boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f = \bigcap\{\, (E \boxed{\scriptstyle k} E, F, g) : \exists\, f\, ;\, \tilde f = (E, F, f) \in \mathrm{scgVS}(\boldsymbol{R})^{\times 2.} \times \mathbf{U}$
and $f \in F^{/E}$ and $\mathrm{dom}\, f \in \tau_{\mathrm{rd}} E$ and $g = \{\, (x, u, y) : x \in \mathrm{dom}\, f$ and
$u \in \upsilon_s E$ and $y \in \upsilon_s F$ and $\forall\, V \in \mathcal{N}_{bh}(y, \tau_{\mathrm{rd}} F)\, ;\, \exists\, \delta \in I\!\!R^+\, ;\, \forall\, t \in I\!\!R\, ;$
$0 < |t| < \delta \Rightarrow t^{-1}(f\text{`}(x + t\,u) - f\text{`}x) \in V\, \}\}$,

(2) $\boldsymbol{\delta}^{\,k}_{\mathrm{Se}}\, \tilde f = \bigcap\{\, (\prod_{\mathrm{cgVS}\, I\!\!R} (k^+ \times \{E\}), F, g) : k : k \in I\!\!N_{\mathrm{o}}$ and
$E, F \in \mathrm{scgVS}(\boldsymbol{R})$ and $\exists\, f, \boldsymbol{f}\, ;\, f \in F^{/E}$ and $\tilde f = (E, F, f)$ and
$(\emptyset, f \circ \langle\, \langle x \rangle : x = x\, \rangle^{-\iota}), (k, g) \in \boldsymbol{f} \in \mathbf{U}^{\,k+1.}$ and $\forall\, i \in k\, ;$
$\boldsymbol{f}\text{`}i^+ = \{\, (\langle x \rangle \hat{\ } \boldsymbol{u} \hat{\ } \langle u \rangle, y) : \forall\, h\, ;\, h = \{\, (x^1, y^1) : (\langle x^1 \rangle \hat{\ } \boldsymbol{u}, y^1) \in \boldsymbol{f}\text{`}i\,\}$
$\Rightarrow (x, u, y) \in \tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}(E, F, h)\,\}\}$,

(3) $\mathcal{C}_{\mathrm{Se}\,0} = \{\, (E, F, f) : f \in F^{/E}$ and $f^{-\iota}\text{```}\tau_{\mathrm{rd}} F \subseteq \tau_{\mathrm{rd}} E\,\} \,|\, \mathrm{scgVS}(\boldsymbol{R})^{\times 2.}$,

any $\tilde f$ with $\mathrm{dom}\, \tau_{\mathrm{rd}} \tilde f \times \upsilon_s \sigma^2_{\mathrm{rd}} \tilde f \subseteq \mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f \neq \mathbf{U}$ we say to be *directionally Seip differentiable*. Any $\tilde f$ with $\tilde f, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f \in \mathcal{C}_{\mathrm{Se}\,0}$ and $\mathrm{dom}\, \tau_{\mathrm{rd}} \tilde f \times \upsilon_s \sigma^2_{\mathrm{rd}} \tilde f \subseteq \mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f$ we say to be *Seip differentiable*. We say that $\tilde f$ is *order k simply Seip – differentiable* if and only if we have $k \in \infty^+$ and for all $i \in k + 1.$ it holds that $\boldsymbol{\delta}^{\,i}_{\mathrm{Se}}\, \tilde f \in \mathcal{C}_{\mathrm{Se}\,0}$ and $\{\, \boldsymbol{x} : \boldsymbol{x}\text{`}\emptyset \in \mathrm{dom}\, \tau_{\mathrm{rd}} \tilde f\,\} \cap (\upsilon_s \sigma^2_{\mathrm{rd}} \tilde f\,)^{i+1.} \subseteq \mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,i}_{\mathrm{Se}}\, \tilde f$. We further let

(4) $\mathsf{D}_{\mathrm{Se}}\, \tilde f = \bigcap\{\, \tilde f' : \exists\, E, F\, ;\, \tilde f$ is Seip differentiable and
$(E, F) = \sigma_{\mathrm{rd}} \tilde f$ and $\tilde f' = (E, \mathcal{L}_{\mathrm{cg}}(E, F), (\tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f\,)^{\vee})\,\}$,

(5) $\mathcal{D}^k_{\mathrm{Seip}} = \{\, \tilde f : k \in \infty^+$ and $\exists\, \boldsymbol{f};\, (\emptyset, \tilde f\,) \in \boldsymbol{f} \in {\mathcal{C}_{\mathrm{Se}\,0}}^{k+1.}$ and
$\forall\, i \in k\, ;\, \boldsymbol{f}\text{`}i$ is Seip differentiable and $\boldsymbol{f}\text{`}i^+ = \boldsymbol{\delta}_{\mathrm{Se}}(\boldsymbol{f}\text{`}i)\,\}$,

and also ${}^{\mathrm{cg}}\bar\Delta\, \tilde f = \mathcal{C}_{\mathrm{Se}\,0}\text{-}\bar\Delta_{\boldsymbol{R}}\, \tilde f$, and suggest that $\boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f$ and $\boldsymbol{\delta}^{\,k}_{\mathrm{Se}}\, \tilde f$ may be referred to by the phrases the Seip – *variation* and the order k Seip – variation of $\tilde f$. The class $\mathsf{D}_{\mathrm{Se}}\, \tilde f$ may be called the Seip *derivative* of $\tilde f$.

Note that any directionally Seip differentiable class $\tilde f$ necessarily is a real vector map since by our definitions from $\mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde f \neq \mathbf{U}$ it follows that there are E, F, f with $E, F \in \mathrm{scgVS}(\boldsymbol{R})$ and $f \in F^{/E}$ and $\tilde f = (E, F, f)$.

An exercise in set theory and logic shows that we have either $\boldsymbol{\delta}^{\,k}_{\mathrm{Se}}\, \tilde f = \mathbf{U}$ or

(u) $k \in I\!N_{\rm o}$ and there are unique $\boldsymbol{g} \in \mathsf{U}^{\,k+1.}$ and $E, F \in \mathrm{scgVS}(\boldsymbol{R})$ and $f \in F^{/E}$ with $\tilde f = (E, F, f)$ and $\boldsymbol{g}`\emptyset = \{ (\langle x\rangle, y) : (x, y) \in f \}$ and $\boldsymbol{g}`i$ a function for all $i \in k^+$ such that in the case $i \neq k$ we have $\{ x : \langle x\rangle\hat{\ }\boldsymbol{x} \in \mathrm{dom}\,(\boldsymbol{g}`i) \} \in \tau_{\rm rd} E$ for every $\boldsymbol{x}$, and for all $\boldsymbol{z}, y$ we have $(\boldsymbol{z}, y) \in \boldsymbol{g}`i^+$ if and only if there are $x, u \in \upsilon_s E$ and $\boldsymbol{x} \in (\upsilon_s E)^{\,i}$ with $\boldsymbol{z} = \langle x\rangle\hat{\ }\boldsymbol{x}\hat{\ }\langle u\rangle$ and

$$y = \tau_{\rm rd} F\text{-}\lim_{t\to 0} t^{-1}(\boldsymbol{g}`i`(\langle x + t\,u\rangle\hat{\ }\boldsymbol{x}) - \boldsymbol{g}`i`(\langle x\rangle\hat{\ }\boldsymbol{x})) \neq \mathsf{U}\,.$$

Here we understand that a limit equals U if it "does not exist". In the case (u) we also have $\boldsymbol{\delta}^{\,k}_{\rm Se}\tilde f = (E^{\,k+1.]_{\rm kvs}}, F, \boldsymbol{g}`k)$, for $E^{\,I\,]_{\rm kvs}} = \prod_{\mathrm{cgVS}\,\mathrm{dom}^2\tau\sigma_{\rm rd} E}(I \times \{E\})$.

43 Proposition. $\mathcal{C}_{\rm Se\,0}$ *is a* BGN *– class on* $\mathrm{scgVS}(\boldsymbol{R})$ *over* $\boldsymbol{R}$.

Proof. First, to see that $\mathcal{C}_{\rm Se\,0}$ is a productive class on $\mathrm{scgVS}(\boldsymbol{R})$ over $\boldsymbol{R}$, we must verify that for all $E, F \in \mathrm{scgVS}(\boldsymbol{R})$ there is G with $\mathcal{C}_{\rm Se\,0}\text{-}\,\mathrm{prod}_{\rm mcl}(E, F, G)$ as specified in Definitions 4 above. To verify this for $G = E \boxtimes F$, first note that by Corollary 27 we have $G \in \mathrm{scgVS}(\boldsymbol{R})$, and then that for $(H, E, f), (H, F, g) \in \mathcal{C}_{\rm Se\,0}$ we have $(H, G, [\,f, g\,]) \in \mathcal{C}_{\rm Se\,0}$ by Lemmas 17 and 15.

For the verification of the postulates (1), ... (6) in Definitions 4, first note that by Lemma 5 we get (6) directly from the definitions since $\mathcal{C}_{\rm Se\,0}\text{-}\,\mathrm{P_{rod}}`(\boldsymbol{R}, E) = \boldsymbol{R} \boxtimes E$ and trivially $\tau_{I\!R} \underset{\rm t}{\times} \tau_{\rm rd} E \subseteq \tau_{I\!R} \underset{\rm k}{\times} \tau_{\rm rd} E = \tau_{\rm rd}(\boldsymbol{R} \boxtimes E)$ when $E \in \mathrm{scgVS}(\boldsymbol{R})$. We have (1), ... (5) as trivialities or well-known facts given by straighforward verifications, noting e.g. that (1) just means that a function is continuous with open domain if it can be expressed as a union of such, and that for (2) under the premise we have $(g \circ f)^{-\iota}``\tau_{\rm rd} G \subseteq f^{-\iota}``(g^{-\iota}``\tau_{\rm rd} G) \subseteq f^{-\iota}``\tau_{\rm rd} F \subseteq \tau_{\rm rd} E$. $\square$

In [17; Theorem 3.8, pp. 82–83], Seip incidentally established the following

44 Lemma. *Let* $\tilde f = (E, F, f)$. *If* $\tilde f$ *is directionally Seip – differentiable and* $\boldsymbol{\delta}_{\rm Se}\tilde f$ *is continuous, there is* g *with the property that* $(E \boxtimes E \boxtimes \boldsymbol{R}, F, g) \in \mathcal{C}_{\rm Se\,0}$, *and also such that for all* x, u, t, y_1 *with* $y_1 \in \upsilon_s F$ *it holds that*

$$f`(x + t\,u) = f`x + t\,y_1 \neq \mathsf{U} \Leftrightarrow \exists\, y\,;\ (x, u, t, y) \in g \text{ and } [\ t \neq 0 \Rightarrow y = y_1\,]\,.$$

Putting $\mathcal{D}^1_{\rm Seip\,72} = \{ (E, F, f) : f$ is *stetig differenzierbar* $E \supset \mathrm{dom}\, f \to F$ in the sense of [16; Definition 4.2, p. 60] $\}$, we have

45 Corollary. *For all* $\tilde f$ *and* $\tilde g$ *it holds that*

(a) $\mathcal{D}^{1.}_{\rm Seip} = \mathcal{D}^{1.}_{\rm BGN}(\mathcal{D}^{0.}_{\rm Seip}, \boldsymbol{R}) = \mathcal{D}^1_{\rm Seip\,72}$,

(b) $\tilde f$ *is Seip – differentiable* $\Rightarrow$ $\mathsf{D}_{\rm Se}\tilde f$ *is a vector map*,

(c) $\tilde f$ *is Seip – differentiable* $\Leftrightarrow$ $\tilde f \in \mathcal{D}^{1.}_{\rm Seip}$ $\Leftrightarrow$
$\tilde f$ *is directionally Seip – differentiable and* $\boldsymbol{\delta}_{\rm Se}\tilde f \in \mathcal{C}_{\rm Se\,0}$,

(d) $\tilde f$ *and* $\tilde g$ *are Seip – differentiable and* $\tau\sigma_{\rm rd}\tilde f = \sigma^2_{\rm rd}\tilde g$
$\Rightarrow$ $\tau_{\rm rd}\,\boldsymbol{\delta}_{\rm Se}(\tilde g \circ \tilde f) = \tau_{\rm rd}\,\boldsymbol{\delta}_{\rm Se}\tilde g \circ [\,\tau_{\rm rd}\tilde f \circ \mathrm{pr}_1, \tau_{\rm rd}\,\boldsymbol{\delta}_{\rm Se}\tilde f\,]$.

Proof. The equality $\mathcal{D}^{1.}_{\rm Seip} = \mathcal{D}^{1.}_{\rm BGN}(\mathcal{D}^{0.}_{\rm Seip}, \boldsymbol{R})$ follows from Lemma 44 above by a nice exercise in logic. For (b) assuming that $\tilde f = (E, F, f) \in \mathcal{D}^{1.}_{\rm Seip}$, it suffices for each fixed $x \in \mathrm{dom}\, f$ to establish linearity $\sigma_{\rm rd} E \to \sigma_{\rm rd} F$ of $\tau_{\rm rd}\,\boldsymbol{\delta}_{\rm Se}\tilde f(x, \cdot)$. This follows from [2; Proposition 2.2, p. 223].

To get $\mathcal{D}^{1.}_{\rm BGN}(\mathcal{D}^{0.}_{\rm Seip}, \boldsymbol{R}) = \mathcal{D}^1_{\rm Seip\,72}$, we first note that for $\tilde f = (E, F, f)$ we have $\tilde f \in \mathcal{D}^1_{\rm Seip\,72}$ if and only if $E, F \in \mathrm{scgVS}(\boldsymbol{R})$ and for $L = \mathcal{L}_{\rm cg}(E, F)$ and $U = \mathrm{dom}\, f$ and $U_3 = \{ (x, u, t) : x, x + t\,u \in U \}$, we have $U \in \tau_{\rm rd} E$ and $f \in (\upsilon_s F)^{\,U}$, and there are $f_1 \in (\upsilon_s L)^{\,U}$ and $h \in (\upsilon_s F)^{\,U_3}$ such that f_1 is continuous $\tau_{\rm rd} E \to \tau_{\rm rd} L$

and h is continuous $\tau_{\mathrm{rd}}(E \circledast E \circledast \boldsymbol{R}) \to \tau_{\mathrm{rd}} F$ with $U \times (\upsilon_s E) \times \{0\} \times \{\mathbf{0}_F\} \subseteq h$, and such that for $(x, u, t) \in U_3$ it holds that

$$f\,\grave{}\,(x + t\,u) = f\,\grave{}\,x + f_1\,\grave{}\,x\,\grave{}\,(t\,u) + t\,h\,\grave{}\,(x, u, t)\,.$$

Noting that by Proposition 31 continuity of (E, L, f_1) is equivalent to that of $(E \circledast E, F, f_1{}^{\wedge})$, and recalling the above established property that $\tilde{f} \in \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$ implies linearity of $(\sigma_{\mathrm{rd}} E, \sigma_{\mathrm{rd}} F, \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f}(x, \cdot))$ for all $x \in U$, it is a simple exercise in logic (to the reader) to verify that $\mathcal{D}^{1\cdot}_{\mathrm{BGN}}(\mathcal{D}^{0\cdot}_{\mathrm{Seip}}, \boldsymbol{R}) = \mathcal{D}^{1}_{\mathrm{Seip}\,72}$.

For (c) assuming that $\tilde{f}$ is directionally Seip–differentiable with $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{C}_{\mathrm{Se}\,0}$, we only have to prove continuity of $\tilde{f}$. Taking $t = 1$ in Lemma 44 above, we have $f\,\grave{}\,(x + u) = f\,\grave{}\,x + g\,\grave{}\,(x, u, 1)$ for u close to $\mathbf{0}_E$, whence the assertion. For (d) we refer the reader to see [2; Proposition 3.1, p. 225]. $\square$

Property (d) in Corollary 45 may be called the *first order chain rule*. The differentiability classes $\mathcal{D}^{k}_{\mathrm{Seip}}$ for different k are related to one another via the recursion rules given in the following

46 Proposition. *For all* $k, \tilde{f}$ *it holds that* $\tilde{f} \in \mathcal{D}^{k+1\cdot}_{\mathrm{Seip}} \Leftrightarrow$ (a) $\Leftrightarrow$ (b), *when the conditions are as given below. It also holds that* $\mathcal{D}^{\infty}_{\mathrm{Seip}} = \{\,\tilde{f} : \forall\, k \in \mathbb{N}_{\mathrm{o}}\,;\ \tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}\,\}$.

(a) $\tilde{f} \in \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$ *and* $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}$,

(b) $\tilde{f}$ *is directionally Seip – differentiable and* $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}$.

Proof. One obtains $\tilde{f} \in \mathcal{D}^{k+1\cdot}_{\mathrm{Seip}} \Leftrightarrow$ (a) similarly as in the proof of Proposition 10 above. For (a) $\Leftrightarrow$ (b), it suffices to observe that $\tilde{f} \in \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$ follows from directional differentiability and continuity of the variation by Corollary 45 (c) above.

For $\mathcal{D}_0 = \{\,\tilde{f} : \forall\, k \in \mathbb{N}_{\mathrm{o}}\,;\ \tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}\,\}$, to establish $\mathcal{D}^{\infty}_{\mathrm{Seip}} = \mathcal{D}_0$, note that we trivially have $\mathcal{D}^{\infty}_{\mathrm{Seip}} \subseteq \mathcal{D}_0$. To get the converse, arbitrarily fixing $\tilde{f} \in \mathcal{D}_0$, and considering the class

$$\Gamma = \{\,\boldsymbol{f} : \exists\, k \in \infty^{+}\,;\ \forall\, i\,;\ (\emptyset, \tilde{f}) \in \boldsymbol{f} \in \mathcal{C}_{\mathrm{Se}\,0}{}^{k+1\cdot} \text{ and } [\ i \in k \Rightarrow \boldsymbol{f}\,\grave{}\,i \text{ is Seip differentiable and } \boldsymbol{f}\,\grave{}\,(i^{+}) = \boldsymbol{\delta}_{\mathrm{Se}}(\boldsymbol{f}\,\grave{}\,i)\ ]\,\},$$

we first see for $\boldsymbol{f}, \boldsymbol{g} \in \Gamma$ that either $\boldsymbol{f} \subseteq \boldsymbol{g}$ or $\boldsymbol{g} \subseteq \boldsymbol{f}$, and consequently that $\bigcup \Gamma$ is a function. In view of $\tilde{f} \in \mathcal{D}_0$, this further gives $\bigcup \Gamma \in \Gamma$ with $\operatorname{dom} \bigcup \Gamma = \mathbb{N}_{\mathrm{o}}$, whence $\tilde{f} \in \mathcal{D}^{\infty}_{\mathrm{Seip}}$ follows directly by definition. $\square$

Some further basic properties of the classes $\mathcal{D}^{k}_{\mathrm{Seip}}$ are given in the following

47 Proposition. *For all* E, F, G, H, P, f, g, k *it holds that*

(1) $\{\,(E, F, U \times \{y\}) : U \in \tau_{\mathrm{rd}} E \text{ and } y \in \upsilon_s F\,\}\,|\,\mathrm{scgVS}(\boldsymbol{R})^{\times 2\cdot} \subseteq \mathcal{D}^{\infty}_{\mathrm{Seip}}$,

(2) $\{\,(E, F, \ell) : \ell \in \upsilon_s \mathcal{L}_{\mathrm{cg}}(E, F)\,\}\,|\,\mathrm{scgVS}(\boldsymbol{R})^{\times 2\cdot} \subseteq \mathcal{D}^{\infty}_{\mathrm{Seip}}$,

(3) ${}^{\mathrm{bi}}\mathcal{L}_{\mathrm{scgVS}}(\boldsymbol{R}) \subseteq \mathcal{D}^{\infty}_{\mathrm{Seip}}$,

(4) $(E, F, f), (E, G, g) \in \mathcal{D}^{k}_{\mathrm{Seip}} \Rightarrow (E, F \circledast G, [\,f, g\,]) \in \mathcal{D}^{k}_{\mathrm{Seip}}$,

(5) $(E, F, f), (F, G, g) \in \mathcal{D}^{k}_{\mathrm{Seip}} \Rightarrow (E, G, g \circ f) \in \mathcal{D}^{k}_{\mathrm{Seip}}$, (chain rule)

(6) $(E, F, f), (H, G, g) \in \mathcal{D}^{k}_{\mathrm{Seip}} \Rightarrow (E \circledast H, F \circledast G, f \underset{2}{\times} g) \in \mathcal{D}^{k}_{\mathrm{Seip}}$,

(7) $P = (E, F) \in \mathrm{scgVS}(\boldsymbol{R})^{\times 2\cdot}$ *and* f *is a function and* $[\ \forall\, Z\,;\ \exists\, h\,;\ Z \in f \Rightarrow Z \in h \subseteq f$ *and* $(P, h) \in \mathcal{D}^{k}_{\mathrm{Seip}}\ ] \Rightarrow (P, f) \in \mathcal{D}^{k}_{\mathrm{Seip}}$.

Proof. We get (1) by a trivial induction using Proposition 46 above when for fixed $\tilde{f} = (E, F, U \times \{y\})$ we note that $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} = (E \circledast E, F, U \times (\upsilon_s E) \times \{\mathbf{0}_F\})$. We get (2) similarly, noting that $\boldsymbol{\delta}_{\mathrm{Se}}(E, F, \ell) = (E \circledast E, F, \ell \circ \mathrm{pr}_2\,|\,(\upsilon_s E)^{\times 2\cdot})$. In order to establish (3), note that for $\tilde{b} = (E_2, F, b) \in {}^{\mathrm{bi}}\mathcal{L}_{\mathrm{scgVS}}(\boldsymbol{R})$ we have

$$\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{b} = \langle\, b\,`(x\,,v) + b\,`(u\,,y) : W = (x\,,y\,;u\,,v) \in (\upsilon_s E_2)^{\times 2.}\,\rangle\,,$$

and hence that also $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{b} \in {}^{\mathrm{bi}}\mathcal{L}_{\mathrm{scgVS}}(\boldsymbol{R})$.

For (4) the case $k=\emptyset$ follows from Proposition 43 above. For the inductive step, letting $f_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(E\,,F\,,f)$ and $g_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(E\,,G\,,g)$, note that we have $\boldsymbol{\delta}_{\mathrm{Se}}(E\,,F \boxtimes G,[\,f\,,g\,]) = (E \boxtimes E\,,F \boxtimes G,[\,f_1\,,g_1\,])$.

For (5) the case $k=\emptyset$ is got from Proposition 43. For $k \in I\!N$, one proceeds by induction, noting that with $f_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(E\,,F\,,f)$ and $g_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(F\,,G\,,g)$, we have $\boldsymbol{\delta}_{\mathrm{Se}}(E\,,G\,,g \circ f) = (E \boxtimes E\,,G\,,g_1 \circ [\,f \circ \mathrm{pr}_1\,,f_1\,])$ by the first order chain rule. For the inductive step, assuming that we have the result with $k \in I\!N_{\mathrm{o}}$ and that $(E\,,F\,,f)\,,(F\,,G\,,g) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$, first note that $(E \boxtimes E\,,F \boxtimes F,[\,f \circ \mathrm{pr}_1\,,f_1\,]) \in \mathcal{D}^{k}_{\mathrm{Seip}}$ by (2) and (4), and then use the inductive assumption again.

Noting that $f \underset{2}{\times} g = [\,f \circ \mathrm{pr}_1\,,g \circ \mathrm{pr}_2\,]$, we get (6) from (2) and (4) and (5).

Finally, for the inductive proof of (7) the case $k=\emptyset$ has been "explained" in the proof of Proposition 43 above. Now assuming that we have (7) for a fixed $k \in I\!N_{\mathrm{o}}$ and for all P,E,F,f, and that also $P=(E\,,F) \in \mathrm{scgVS}(\boldsymbol{R})^{\times 2.}$ and f is a function such that $(*)$ $[\ \forall\, Z\,;\, \exists\, h\,;\, Z \in f \Rightarrow Z \in h \subseteq f$ and $(P,h) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}\ ]$ holds, to establish $(P,f) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$, by Proposition 46(b) it suffices that (P,f) is directionally Seip–differentiable and that $\boldsymbol{\delta}_{\mathrm{Se}}(P,f) \in \mathcal{D}^{k}_{\mathrm{Seip}}$.

Having (P,h) directionally Seip–differentiable when $(P,h) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}} \subseteq \mathcal{D}^{1.}_{\mathrm{Seip}}$, inspection of the definitions shows that from $(*)$ we get that (P,f) is directionally Seip–differentiable. To get $\boldsymbol{\delta}_{\mathrm{Se}}(P,f) \in \mathcal{D}^{k}_{\mathrm{Seip}}$, putting $G = E \boxtimes E$ and $Q = (G,F)$ and $f_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(P,f)$, we apply the inductive assumption with $Q\,,G,F,f_1$ in place of P,E,F,f. Indeed, if $W \in f_1$, there are $x\,,v$ with $W = (x\,,v\,,f_1`(x\,,v))$. For $Z = (x\,,f\,`x)$ then $Z \in f$ holds whence by $(*)$ there is h with $Z \in h \subseteq f$ and $(P,h) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$. For $h_1 = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}(P,h)$ it follows (by an exercise to the reader) that $W \in h_1 \subseteq f_1$ with $(Q\,,h_1) \in \mathcal{D}^{k}_{\mathrm{Seip}}$. Consequently, the premise in the inductive assumption holds, hence also the conclusion, that is $\boldsymbol{\delta}_{\mathrm{Se}}(P,f) = (Q\,,f_1) \in \mathcal{D}^{k}_{\mathrm{Seip}}$. $\square$

48 Proposition. *For all k it holds that*

$$\mathcal{D}^{k}_{\mathrm{Seip}} = \{\, \tilde{f} : \tilde{f} \text{ is order } k \text{ simply Seip–differentiable} \,\}\,.$$

Proof. Let $\mathcal{D}_k = \{\, \tilde{f} : \tilde{f}$ is order k simply Seip–differentiable $\}$, and let $(k)_{\mathrm{A}}$ mean that $\mathcal{D}^{k}_{\mathrm{Seip}} = \mathcal{D}_k$ holds. From Definitions 42 above, one first checks that we have $\mathcal{D}_k = \emptyset$ for $k \notin \infty^{+}$, and that $\mathcal{D}_\infty = \{\, \tilde{f} : \forall\, k \in I\!N_{\mathrm{o}}\,;\, \tilde{f} \in \mathcal{D}_k \,\}$. In view of Proposition 46 above, it hence suffices to establish $\forall\, k \in I\!N_{\mathrm{o}}\,;\,(k)_{\mathrm{A}}$.

Given $\tilde{f} = (E\,,F\,,f) \in \mathrm{scgVS}(\boldsymbol{R})^{\times 2.} \times \mathbf{U}$ with $f \in F^{/E}$, and considering the linear homeomorphisms $\iota_1 : E \to E^{\,1.]_{\mathrm{kvs}}}$ and $\iota_2 : E \boxtimes E \to E^{\,2.]_{\mathrm{kvs}}}$ given by $x \mapsto \langle x \rangle$ and $(x\,,y) \mapsto \langle\, x\,,y\,\rangle$, respectively, from the result around (u) after Definitions 42 we see that $f = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}^{\,0.}_{\mathrm{Se}}\,\tilde{f} \circ \iota_1$ and $[\ \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}^{\,1.}_{\mathrm{Se}}\,\tilde{f} \circ \iota_2$ or $[\ \mathrm{dom}\, f \notin \tau_{\mathrm{rd}}\, E$ and $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} = \boldsymbol{\delta}^{\,1.}_{\mathrm{Se}}\,\tilde{f} = \mathbf{U}\ ]\ ]$. Using this, by Definitions 42 we get $\forall\, k \in \mathbf{2.}\,;\,(k)_{\mathrm{A}}$.

We next assume that $k \in I\!N$ with $(k)_{\mathrm{A}}$, and proceed to get $(k^{+})_{\mathrm{A}}$ as follows. First letting $\tilde{f} = (E\,,F\,,f) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$, then $\tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}$, and by Proposition 46(a) also $\tilde{f} \in \mathcal{D}^{1.}_{\mathrm{Seip}}$ and $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}^{k}_{\mathrm{Seip}}$. Using $(k)_{\mathrm{A}}$, we obtain $\tilde{f}\,,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}_k$. A "straightforward" induction on $l \in k^{+}$ shows that we have

$$\tau_{\mathrm{rd}}\,\boldsymbol{\delta}^{\,l+1.}_{\mathrm{Se}}\,\tilde{f}\,`\boldsymbol{x} = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f}\,`\big(\langle(\boldsymbol{x}`\emptyset\,,\boldsymbol{x}`1.)\rangle\,\hat{}\,\langle\,(\boldsymbol{x}`i^{++},\mathbf{0}_E) : i \in l\,\rangle\big)\,,$$

for $\boldsymbol{x} \in (\upsilon_s E)^{\,l+2.}$ with $\boldsymbol{x}`\emptyset \in \mathrm{dom}\, f$. Taking here $l=k$, we get $\boldsymbol{\delta}^{\,k+1.}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{C}_{\mathrm{Se}\,0}$, and see that $\{\, \boldsymbol{x} : \boldsymbol{x}`\emptyset \in \mathrm{dom}\, f \,\} \cap (\upsilon_s E)^{\,k+2.} \subseteq \mathrm{dom}\ \tau_{\mathrm{rd}}\,\boldsymbol{\delta}^{\,k+1.}_{\mathrm{Se}}\,\tilde{f}$. Consequently, noting $\tilde{f} \in \mathcal{D}_k$, we get $\tilde{f} \in \mathcal{D}_{k+1.}$.

For the converse proof we first introduce the notation

$$\mathrm{R}\, x\, \boldsymbol{u}\, i\, \boldsymbol{x} = \langle\, x\,, \boldsymbol{u}\grave{}\, i\,\rangle\hat{}\,(\boldsymbol{x}\,|\,i)\hat{}\big(\boldsymbol{x}\circ\langle\, i+j+1\boldsymbol{.} : j\in I\!N_{\mathrm{o}}\rangle\big)\,.$$

Next, using the linearity from the proof of (b) in Corollary 45 above, by induction on $l \in I\!N_{\mathrm{o}}$ one (= the reader!) proves ($*$) that for an order l^{+} simply Seip – differentiable map $\tilde{f} = (E\,, F\,, f\,)$ and for any $\boldsymbol{x} \in (\upsilon_s E\,)^{\,l}$ and $x \in \mathrm{dom}\, f$ we have a linear map $\langle\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,(\langle\, x\,, u\,\rangle\hat{}\,\boldsymbol{x}) : u \in \upsilon_s E\,\rangle : \sigma_{\mathrm{rd}}\, E \to \sigma_{\mathrm{rd}}\, F$.

Now, assuming that $\tilde{f} = (E\,, F\,, f\,) \in \mathcal{D}_{k+1\boldsymbol{.}}$, then $\tilde{f} \in \mathcal{D}_{1\boldsymbol{.}}$ whence by $(1\boldsymbol{.})_{\mathrm{A}}$ we have $\tilde{f} \in \mathcal{D}^{1\boldsymbol{.}}_{\mathrm{Seip}}$. Consequently, by Proposition 46 (a) and $(k)_{\mathrm{A}}$ to get $\tilde{f} \in \mathcal{D}^{k+1\boldsymbol{.}}_{\mathrm{Seip}}$ it suffices to establish $\boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f} \in \mathcal{D}_k$. To get this, by induction on $l \in k^{+}$ one proves (x) that for $x \in \mathrm{dom}\, f$ and $u \in \upsilon_s E$ and $\boldsymbol{x}, \boldsymbol{u} \in (\upsilon_s E\,)^{\,l}$ we have

$$\begin{aligned}
&\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\,(x\,, u)\,\rangle\hat{}\,[\,\boldsymbol{x}\,, \boldsymbol{u}\,]\big) = \\
&\qquad\qquad \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x\,, u\,\rangle\hat{}\,\boldsymbol{x}\big) + \textstyle\sum_{\,i\in l}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\, x\, \boldsymbol{u}\, i\, \boldsymbol{x}\,.
\end{aligned}$$

For the inductive step here, noting ($*$) above, and also letting $y\,, v \in \upsilon_s E$, one checks the computation

$$\begin{aligned}
&\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\,(x\,, u)\,\rangle\hat{}\,[\,\boldsymbol{x}\,, \boldsymbol{u}\,]\hat{}\,\langle\,(y\,, v)\,\rangle\big) \\
&\qquad = \tau_{\mathrm{rd}}\, F\text{-}\lim_{t\to 0}\, t^{-1}\big(\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\,(x+t\,y\,, u+t\,v)\,\rangle\hat{}\,[\,\boldsymbol{x}\,, \boldsymbol{u}\,]\big) \\
&\qquad\qquad\qquad - \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\,(x\,, u)\,\rangle\hat{}\,[\,\boldsymbol{x}\,, \boldsymbol{u}\,]\big)\big) \\
&= \tau_{\mathrm{rd}}\, F\text{-}\lim_{t\to 0}\, t^{-1}\big(\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x+t\,y\,, u+t\,v\,\rangle\hat{}\,\boldsymbol{x}\big) - \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x\,, u\,\rangle\hat{}\,\boldsymbol{x}\big) \\
&\qquad\qquad + \textstyle\sum_{\,i\in l}\, \big(\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\,(x+t\,y)\, \boldsymbol{u}\, i\, \boldsymbol{x} - \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\, x\, \boldsymbol{u}\, i\, \boldsymbol{x}\big)\big) \\
&= \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+2\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x\,, u\,\rangle\hat{}\,\boldsymbol{x}\hat{}\,\langle\, y\,\rangle\big) + \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x\,, v\,\rangle\hat{}\,\boldsymbol{x}\big) \\
&\qquad\qquad + \textstyle\sum_{\,i\in l}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\, x\, \boldsymbol{u}\, i\,(\boldsymbol{x}\hat{}\,\langle\, y\,\rangle) \\
&= \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+2\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle\, x\,, u\,\rangle\hat{}\,\boldsymbol{x}\hat{}\,\langle\, y\,\rangle\big) + \textstyle\sum_{\,i\in l+1\boldsymbol{.}}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l+1\boldsymbol{.}}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\, x\,(\boldsymbol{u}\hat{}\,\langle\, v\,\rangle)\, i\,(\boldsymbol{x}\hat{}\,\langle\, y\,\rangle)\,.
\end{aligned}$$

Finally, from (x) we "directly" see that $\boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f} \in \mathcal{C}_{\mathrm{Se}\,0}$ with also

$$\{\, \boldsymbol{z} : \boldsymbol{z}\grave{}\,\emptyset \in \mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}\,\} \cap (\upsilon_s(E \circledast E\,))^{\,l+1\boldsymbol{.}} \subseteq \mathrm{dom}\, \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f}$$

for all $l \in k^{+}$. Consequently $\boldsymbol{\delta}_{\mathrm{Se}}\, \tilde{f} \in \mathcal{D}_k$, as was to be established. ☐

A not quite short analytic-combinatorial proof shows that for an order l simply Seip – differentiable map $\tilde{f} = (E\,, F\,, f\,)$ and for $x \in \mathrm{dom}\, f$ and $\boldsymbol{x} \in (\upsilon_s E\,)^{\,l}$ and any bijection $\sigma : l \to l$ we have $\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle x\rangle\hat{}\,\boldsymbol{x}\big) = \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle x\rangle\hat{}\,(\boldsymbol{x}\circ\sigma)\big)$. If we had first established this result, in the above proof we could have written also

$$\tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\mathrm{R}\, x\, \boldsymbol{u}\, i\, \boldsymbol{x} = \tau_{\mathrm{rd}}\, \boldsymbol{\delta}^{\,l}_{\mathrm{Se}}\, \tilde{f}\,\grave{}\,\big(\langle x\rangle\hat{}\,(\boldsymbol{x}\,|\,(\mathbf{U}\setminus\{i\})\cup\{\,(i\,, \boldsymbol{u}\grave{}\, i)\,\})\big)\,.$$

49 *Construction.* With any classes $\tilde{g}\,, k$ we associate

$$\begin{aligned}
&I^{\,k}\tilde{g} = \textstyle\bigcap\big\{\,(E \circledast E\,, F\,, g_1) : k \in I\!N_{\mathrm{o}} \text{ and } E\,, F \text{ are real topologized} \\
&\qquad \text{vector spaces and } g \in F^{\,/E} \text{ and } \tilde{g} = (E\,, F\,, g) \text{ and } g_1 = \big\{\,(x\,, u\,, y) : \\
&\quad [\,\forall\, s\,;\, 0 \le s \le 1 \Rightarrow x + s\,u \in \mathrm{dom}\, g\,] \text{ and } y = F\text{-}\textstyle\int_0^1 s^{\,k} g\grave{}\,(x+s\,u)\,\mathrm{d}\,s\,\big\}\big\}\,.
\end{aligned}$$

By elementary set theoretic manipulations, we have either $I^{\,k}\tilde{g} = \bigcap\emptyset = \mathbf{U}$, or $k \in I\!N_{\mathrm{o}}$ and there are real topologized vector spaces $E\,, F$ and $g \in F^{\,/E}$ with $\tilde{g} = (E\,, F\,, g)$. In the latter case $I^{\,k}\tilde{g} = (E \circledast E\,, F\,, g_1)$ where g_1 is the function defined on the set of the $(x\,, u)$ with $x \in \mathrm{dom}\, g$ and $u \in \upsilon_s E$ such that $x + s\,u \in \mathrm{dom}\, g$ for $0 \le s \le 1$, and such that also the function $s \mapsto s^{\,k} g\grave{}\,(x+s\,u)$ on $[\,0\,,1\,]$ in F is Riemann integrable to some y . Deeper properties important for our purposes of the integration families $J : \mathcal{C}_{\mathrm{Se}\,0} \ni \tilde{g} \mapsto I^{\,k}\tilde{g}$ are given by the following

50 Lemma. *Let* $k \in \mathbb{N}_{\mathrm{o}}$ *and* $J = \langle I^{k}\tilde{g} : \tilde{g} \in \mathcal{C}_{\mathrm{Se}0}\rangle$. *Then* rng $J \subseteq \mathcal{C}_{\mathrm{Se}0} = \operatorname{dom} J$ *and* $J``\mathcal{D}^{1\cdot}_{\mathrm{Seip}} \subseteq \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$ *hold. Also if* $\tilde{g} = (E, F, g) \in \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$, *for all* x, y, u, v *such that* $(x, u) \in \operatorname{dom} \tau_{\mathrm{rd}} I^{k}\tilde{g}$ *and* $y, v \in v_{s}E$ *it holds that* $\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\, I^{k}\tilde{g}`(x, u\,; y, v) =$

$$F\text{-}\int_0^1 (s^{k}\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u, y) + s^{k+1\cdot}\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u, v))\,\mathrm{d}\,s\,.$$

Proof. Write $I = [\,0, 1\,]$. Arbitrarily fixing $\tilde{g} = (E, F, g) \in \mathcal{C}_{\mathrm{Se}0}$, we first show that $I^{k}\tilde{g} \in \mathcal{C}_{\mathrm{Se}0}$. Putting $G = E \boxtimes E$ and $F_1 = \tau_{\mathrm{kv}}`F$ and $g_1 = \tau_{\mathrm{rd}} I^{k}\tilde{g}$ , note that since $\tau_{\mathrm{rd}} G$ is a compactly generated topology, and since we have $F = \kappa_{\mathrm{tv}}`F_1$, it suffices that (G, F_1, g_1) has open domain and is continuous. Hence, arbitrarily fixing $z = (x, u) \in \operatorname{dom} g_1$ and a closed convex $V \in \mathcal{N}_{o} F_1$, it suffices to show (e) existence of $W \in \mathcal{N}_{bh}(z, \tau_{\mathrm{rd}} G)$ such that for all $w \in W$ we have $g_1`w - g_1`z \in V$. Note that this implies $W \subseteq \operatorname{dom} g_1$ since by [13; Theorem 69, p. 261] we have $g_1`w - g_1`z = \mathbf{U} \notin V$ if $w \notin \operatorname{dom} g_1$.

To establish (e) above, we consider the set

$$\begin{aligned} \mathcal{N}_0 = \{\,(W, N) : W \in \mathcal{N}_{bh}(z, \tau_{\mathrm{rd}} G) \text{ and } \exists\, t \in I\,;\ N \in \mathcal{N}_{bh}(t, \tau_{\mathbb{R}}) \text{ and }& \\ \forall\, x^1, u^1, s\,;\ (x + x^1, u + u^1) \in W \text{ and } s \in I \cap N& \\ \Rightarrow g`(x + x^1 + s\,(u + u^1)) - g`(x + s\,u) \in V\,\}\,.& \end{aligned}$$

Putting $m = \langle x + s\,u : X = (x, u, s) \in v_{s}(G \sqcap \boldsymbol{R})\rangle$, note that m is a continuous second order polynomial function $G \sqcap \boldsymbol{R} \to E$ since in its decomposition

$$\begin{aligned} X = (x, u, s) \mapsto (x\,; s, u) = (x, Z) &\mapsto (x, \tau\sigma_{\mathrm{rd}} E`Z) \\ &= (x, v) = Y \mapsto \sigma^{2}_{\mathrm{rd}} E`Y = m`X \end{aligned}$$

besides the continuous linear projections as factors there are the continuous linear and bilinear $(G, E, \sigma^{2}_{\mathrm{rd}} E)$ and $(\boldsymbol{R} \sqcap E, E, \tau\sigma_{\mathrm{rd}} E)$. Consequently, since (E, F_1, g) has open domain and is continuous, the same holds also for $(G \sqcap \boldsymbol{R}, F_1, g \circ m)$.

Using this, we see that $I \subseteq \bigcup \operatorname{rng} \mathcal{N}_0$, whence $\tau_{\mathbb{R}}$−compactness of I gives existence of a finite $\mathcal{N} \subseteq \mathcal{N}_0$ with $I \subseteq \bigcup \operatorname{rng} \mathcal{N}$. Putting $W = \bigcap \operatorname{dom} \mathcal{N}$, we have $W \in \mathcal{N}_{bh}(z, \tau_{\mathrm{rd}} G)$, and for (e) it now remains to verify that for arbitrarily fixed $w = z + (x^1, u^1) \in W$ we have $g_1`w - (g_1`z) \in V$. Letting

$$\gamma = \langle\, s^{k}(g`(x + x^1 + s\,(u + u^1)) - g`(x + s\,u)) : s \in I\,\rangle\,,$$

we have $g_1`w - (g_1`z) = F\text{-}\int_0^1 \gamma = F_1\text{-}\int_0^1 \gamma$, whence by our Lemma 38 it suffices that rng $\gamma \subseteq V$, which in turn directly follows from our arrangement above.

Next assuming that $\tilde{g} \in \mathcal{D}^{1\cdot}_{\mathrm{Seip}}$, we show that for $z = (x, u) \in \operatorname{dom} g_1$ and $w = (y, v)$ with $y, v \in v_{s}E$ we have the asserted variation formula $\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\, I^{k}\tilde{g}`(z, w) = Y$, putting $Y = F\text{-}\int_0^1 (s^{k}\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u, y) + s^{k+1\cdot}\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u, v))\,\mathrm{d}\,s$. Noting that the topologies $\tau_{\mathrm{rd}} F$ and $\tau_{\mathrm{rd}} F_1$ have the same convergent sequences, one quickly deduces that for arbitrarily given closed convex $V \in \mathcal{N}_{o} F_1$, it suffices to show (x) existence of some $\delta \in \mathbb{R}^{+}$ such that for all $t \in \mathbb{R}$ with $0 < |\,t\,| < \delta$ we have $t^{-1}(g_1`(z + t\,w) - g_1`z) - Y \in V$.

To establish (x) above, letting

$$\Gamma\, v_1\, s\, t = \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u + t\,(y + s\,v), v_1) - \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}`(x + s\,u, v_1)\,,$$

for each fixed $v_1 \in v_{s}E$ the function $\Gamma\, v_1 : S = (s, t) \mapsto \Gamma\, v_1\, s\, t$ defined exactly for the $S \in \mathbb{R}^{\times 2\cdot}$ having $x + s\,u + t\,(y + s\,v) \in \operatorname{dom} g$ is continuous $\mathcal{T} = \tau_{\mathbb{R}} \times_{\mathrm{t}} \tau_{\mathbb{R}} \to \tau_{\mathrm{rd}} F_1$ and satisfies $I \times \{0\} \subseteq (\Gamma\, v_1)^{-\iota}``\{\mathbf{0}_{F}\}$ and $\operatorname{dom}(\Gamma\, v_1) \in \mathcal{T}$. Using this in conjunction with compactness of I, one deduces existence of $\delta \in \mathbb{R}^{+}$ such that we have $\Gamma\, v\, s\, t, \Gamma\, y\, s\, t \in \frac{1}{2} V$ for all $s \in I$ and $t \in \mathbb{R}$ with $|\,t\,| < \delta$.

We now arbitrarily fix $t \in \mathbb{R}$ with $0 < |\,t\,| < \delta$, and consider in the space F_1 the curve $c = \langle\, t^{-1}(g_1`(z + s_1\,t\,w) - g_1`z) - s_1 Y : s_1 \in I\,\rangle$. Note that for every $s_1 \in I$ we have $x + s\,u + s_1\,t\,(y + s\,v) \in \operatorname{dom} g$ for all $s \in I$, and also that the function

$s \mapsto g\grave{}(x + s\,u + s_1\,t\,(y + s\,v))$ is continuous. Recalling Proposition 37 above, we hence have $s_1 \in \operatorname{dom} c$, and thus indeed $\operatorname{dom} c = I$. Noting that we have got (x) once $c\grave{}1 \in V$ is established, by the *mean value theorem* (Proposition 40 above) it suffices to show that c is differentiable with $\operatorname{rng} \mathsf{D}_{F_1} c \subseteq V$.

For this arbitrarily fixing $s_1 \in I$, using Proposition 41 and noting the linearity from the proof of (b) in Corollary 45 above, a direct computation gives

$$\mathsf{D}_{F_1} c\grave{}s_1 = F_1\text{-}\int_0^1 (s^k\,\Gamma\,y\,s\,(s_1 t) + s^{k+1.}\,\Gamma\,v\,s\,(s_1 t))\,\mathrm{d}\,s\,,$$

whence by Lemma 38 we get $\mathsf{D}_{F_1} c\grave{}s_1 \in \frac{1}{2}V + \frac{1}{2}V \subseteq V$.

We have now (x) and by the variation formula we also have the decomposition

$$\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,I^k\tilde{g} = \sigma^2_{\mathrm{rd}}\,F \circ ((\tau_{\mathrm{rd}}\,I^k\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}) \underset{2}{\times} (\tau_{\mathrm{rd}}\,I^{k+1.}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g})) \circ \ell\,,$$

where the continuous linear map $\ell : (E \boxed{\scriptstyle k} E) \boxed{\scriptstyle k} (E \boxed{\scriptstyle k} E) = E_4 \to E_4 \boxed{\scriptstyle k} E_4$ is given by $(x\,,u\,;y\,,v) \mapsto (x\,,y\,;u\,,\mathbf{0}_E\,;(x\,,v\,;u\,,\mathbf{0}_E))$. Having already established the inclusion $\operatorname{rng} J \subseteq \mathcal{C}_{\mathrm{Se}0}$, this gives $\boldsymbol{\delta}_{\mathrm{Se}}\,I^k\tilde{g} \in \mathcal{C}_{\mathrm{Se}0}$ when applied also to $k + 1.$ in place of k, and consequently by Corollary 45(c) also $J\,``\,\mathcal{D}^{1.}_{\mathrm{Seip}} \subseteq \mathcal{D}^{1.}_{\mathrm{Seip}}$ is verified. $\square$

In proving that $\mathcal{D}^k_{\mathrm{Seip}} = \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$ we shall utilize the following

51 Corollary. *For any k and $l \in I\!N_{\mathrm{o}}$ and $\tilde{g} \in \mathcal{D}^k_{\mathrm{Seip}}$, it holds that $I^l\tilde{g} \in \mathcal{D}^k_{\mathrm{Seip}}$.*

Proof. Since $\mathcal{D}^k_{\mathrm{Seip}} = \emptyset$ if $k \notin \infty^+$, it suffices to prove by induction that for $k \in I\!N_{\mathrm{o}}$ the assertion holds for all l and $\tilde{g}$. The case $k = \emptyset$ or $k = 1.$ having been settled by Lemma 50 above, assuming that the assertion holds for a fixed $k \in I\!N$, and that $\tilde{g} = (E\,, F\,, g) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$, by Proposition 46 we have $\tilde{g} \in \mathcal{D}^{1.}_{\mathrm{Seip}}$ and $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g} \in \mathcal{D}^k_{\mathrm{Seip}}$. By the inductive assumption then $I^l\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}\,, I^{l+1.}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g} \in \mathcal{D}^k_{\mathrm{Seip}}$, and by Lemma 50 also $I^l\tilde{g} \in \mathcal{D}^{1.}_{\mathrm{Seip}}$ holds together with the decomposition

$$\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,I^l\tilde{g} = \sigma^2_{\mathrm{rd}}\,F \circ ((\tau_{\mathrm{rd}}\,I^l\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g}) \underset{2}{\times} (\tau_{\mathrm{rd}}\,I^{l+1.}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{g})) \circ \ell\,,$$

where the continuous linear map $(E_4\,, E_4 \boxed{\scriptstyle k} E_4\,, \ell)$ is as in the proof of Lemma 50 above. By Proposition 46 and items (2), (6) and (5) of Proposition 47 we hence obtain the membership $I^l\tilde{g} \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$ as required. $\square$

52 Theorem. *For all $k \in \infty^+$ it holds that $\mathcal{D}^k_{\mathrm{Seip}} = \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$.*

Proof. It clearly suffices to treat the cases $k \in \infty = I\!N_{\mathrm{o}}$. Proceeding by induction, we first note that the case $k = \emptyset$ is trivial, and that the case $k = 1.$ has been settled in (a) of Corollary 45 above.

Let now $k \in I\!N$ with $\mathcal{D}^k_{\mathrm{Seip}} = \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$. If $\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$, then $\tilde{f} \in \mathcal{D}^{1.}_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R}) \subseteq \mathcal{D}^{1.}_{\mathrm{Seip}}$ and ${}^{\mathrm{cg}}\bar{\Delta}\,\tilde{f} \in \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R}) \subseteq \mathcal{D}^k_{\mathrm{Seip}}$. Putting $\tilde{\ell} = (E \boxed{\scriptstyle k} E\,, E \boxed{\scriptstyle k} E \boxed{\scriptstyle k} \boldsymbol{R}\,, \ell)$ where $\ell = \langle\,(x\,,u\,,0) : z = (x\,,u) \in (\upsilon_s E)^{\times 2.}\,\rangle$, we have $\tilde{\ell}$ a continuous linear map, hence $\tilde{\ell} \in \mathcal{D}^k_{\mathrm{Seip}}$, and also $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} = {}^{\mathrm{cg}}\bar{\Delta}\,\tilde{f} \circ \tilde{\ell}$, whence the chain rule gives $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}^k_{\mathrm{Seip}}$. Using $\tilde{f} \in \mathcal{D}^{1.}_{\mathrm{Seip}}$ we get $\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$.

To establish the converse, assuming that $\tilde{f} = (E\,, F\,, f) \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$, we have $\tilde{f} \in \mathcal{D}^{1.}_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$, whence to get $\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$, it suffices to obtain ${}^{\mathrm{cg}}\bar{\Delta}\,\tilde{f} \in \mathcal{D}^k_{\mathrm{Seip}} \subseteq \mathcal{D}^k_{\mathrm{BGN}}(\mathcal{D}^{0.}_{\mathrm{Seip}}\,, \boldsymbol{R})$. By Proposition 47(7), considering $W = (x_0\,, u_0\,, t_0\,, y) \in f^1 = \tau_{\mathrm{rd}}\,{}^{\mathrm{cg}}\bar{\Delta}\,\tilde{f}$, with $G = E \boxed{\scriptstyle k} E \boxed{\scriptstyle k} \boldsymbol{R}$, it suffices to establish h with $W \in h \subseteq f^1$ and $(G\,, F\,, h) \in \mathcal{D}^k_{\mathrm{Seip}}$. If $t_0 \neq 0$, we may take $h = f^1\,|\,(\mathrm{pr}_2^{-\iota}\,[\,\mathbf{U} \setminus \{0\}\,])$.

To handle the case $t_0 = 0$, letting $\tilde{m} = (G\,, E \boxed{\scriptstyle k} E \boxed{\scriptstyle k} (E \boxed{\scriptstyle k} E)\,, m)$, where $m = \langle\,(x\,,u\,;t\,u\,,\mathbf{0}_E) : X = (x\,,u\,,t) \in \upsilon_s G\,\rangle$, we have $\tilde{m}$ a continuous second order polynomial map, and hence $\tilde{m} \in \mathcal{D}^k_{\mathrm{Seip}}$. By $\tilde{f} \in \mathcal{D}^{k+1.}_{\mathrm{Seip}}$ we have $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde{f} \in \mathcal{D}^k_{\mathrm{Seip}}$, whence

by Corollary 51 above we get $I^{\,0\cdot}\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f \in \mathcal{D}^{\,k}_{\mathrm{Seip}}$, and consequently for $\tilde h = (G, F, h)$ $= I^{\,0\cdot}\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f \circ \tilde m$ the chain rule gives $\tilde h \in \mathcal{D}^{\,k}_{\mathrm{Seip}}$. For $X = (x\,, u\,, t) \in \operatorname{dom} h$ having

$$\begin{aligned} h`X &= \tau_{\mathrm{rd}}\, I^{\,0\cdot}\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(m`X) = \tau_{\mathrm{rd}}\, I^{\,0\cdot}\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x\,, u\,; t\,u\,, \mathbf{0}_E) \\ &= F\text{-}\!\int_0^1 \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x + s\,t\,u\,, u)\,\mathrm{d}\,s = f^{\,1}`(x\,, u\,, t) = f^{\,1}`X\,, \end{aligned}$$

we see that $W \in h \subseteq f^{\,1}$, completing the inductive step, and the whole proof. $\square$

53 Lemma. *Let* $E\,, F, G \in \operatorname{scgVS}(\boldsymbol{R})$ *and* $L = \mathcal{L}_{\mathrm{cg}}(F, G)$ *and* $U \in \tau_{\mathrm{rd}}\,E$ *and also* $f_1 \in \big((\upsilon_s G)^{\,\upsilon_s F}\big)^{\,U}$ *and* $f = f_1^{\,\wedge}$. *For the condition* (n) *below, for all* $k \in I\!N_{\mathrm{o}}$ *it then holds that* $[\ (\mathrm{n})$ and $(E \boxtimes F, G, f) \in \mathcal{D}^{\,k}_{\mathrm{Seip}}\ ] \Leftrightarrow (E\,, L\,, f_1) \in \mathcal{D}^{\,k}_{\mathrm{Seip}}$.

(n) $f_1`x$ *is linear* $\sigma_{\mathrm{rd}}\,F \to \sigma_{\mathrm{rd}}\,G$ *for every* $x \in U$.

Proof. If $k = \emptyset$, the assertion follows from Proposition 31 above. We next prove the assertion for $k = 1\textbf{.}$. Let $\tilde f = (E \boxtimes F, G, f)$ and $\tilde f_1 = (E\,, L\,, f_1)$. First assuming (n) and that $\tilde f \in \mathcal{D}^{\,1\cdot}_{\mathrm{Seip}}$, to prove that $\tilde f_1 \in \mathcal{D}^{\,1\cdot}_{\mathrm{Seip}}$, by Proposition 46 it suffices that $\tilde f_1$ is directionally Seip$-$differentiable and that $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1$ is continuous.

Putting $g_1 = \langle\,\langle\,\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x\,, v\,; u\,, \mathbf{0}_F) : v \in \upsilon_s F\,\rangle_{\mathrm{old}} : Z = (x\,, u) \in O\,\rangle\,,$ where $O = U \times (\upsilon_s E)\,,$ to establish directional differentiability with $\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1 = g_1\,,$ we arbitrarily fix $Z = (x\,, u) \in O$, and take any $G_0 \in \kappa_{\mathrm{tv}}^{\;-\iota}``\{G\} \cap \operatorname{LCS}(\boldsymbol{R})$. In view of Lemma 34 it then suffices for arbitrarily given $\tau_{\mathrm{rd}}\,F-$compact K and closed convex $V \in \mathcal{N}_o\,G_0$ to show (e) existence of $\delta \in I\!R^+$ such that for all $t \in I\!R$ and $v \in K$ with $0 < |\,t\,| < \delta$ we have $\varDelta(t\,, v) \in V$, upon putting

$$\varDelta(t\,, v) = t^{-1}(f\,`(x + t\,u\,, v) - f\,`(x\,, v)) - \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x\,, v\,; u\,, \mathbf{0}_F)\,.$$

To establish (e), using continuity of $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f$ and of the algebraic operations of E and compactness of K, we first see that there is $\delta \in I\!R^+$ such that

$$(*)\qquad \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x + t\,u\,, v\,; u\,, \mathbf{0}_F) - \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x\,, v\,; u\,, \mathbf{0}_F) \in V$$

whenever $t \in I\!R$ and $v \in K$ with $|\,t\,| < \delta$. To get $\varDelta(t\,, v) \in V$ for arbitrarily fixed such $t\,, v$ with $t \neq 0$, by the mean value theorem considering the differentiable curve c on $[\,0\,, 1\,]$ with values in G_0 and defined by

$$s \mapsto t^{-1}(f\,`(x + s\,t\,u\,, v) - f\,`(x\,, v)) - s\,\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f\,`(x\,, v\,; u\,, \mathbf{0}_F)\,,$$

it suffices that $\operatorname{rng}\,\mathsf{D}_{G_0}\,c \subseteq V$. By $(*)$ this indeed is the case. Having now $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1 = (E \boxtimes E\,, L\,, g_1)\,,$ to prove that $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1$ is continuous, we just note that this directly follows from Proposition 31 by continuity of $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f$.

Next assuming that $\tilde f_1 \in \mathcal{D}^{\,1\cdot}_{\mathrm{Seip}}$, trivially (n) holds, and to prove that $\tilde f \in \mathcal{D}^{\,1\cdot}_{\mathrm{Seip}}$, we again establish directional differentiability and continuity of the variation. With $O_2 = U \times (\upsilon_s F) \times (\upsilon_s E \times (\upsilon_s F))\,,$ letting

$$g = \langle\,\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1`(x\,, u)`v + f\,`(x\,, v_1) : W = (x\,, v\,; u\,, v_1) \in O_2\,\rangle\,,$$

to prove directional differentiability together with the equality $\tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f = g\,,$ for arbitrarily fixed $W = (x\,, v\,; u\,, v_1) \in O_2\,,$ for $0 \neq t \in I\!R$ with $|\,t\,|$ small, having

$$\begin{aligned} &t^{-1}(f\,`(x + t\,u\,, v + t\,v_1) - f\,`(x\,, v)) - \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1`(x\,, u)`v - f\,`(x\,, v_1) \\ &= t^{-1}(f\,`(x + t\,u\,, v) - f\,`(x\,, v)) - \tau_{\mathrm{rd}}\,\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f_1`(x\,, u)`v \\ &\qquad\qquad + f\,`(x + t\,u\,, v_1) - f\,`(x\,, v_1)\,, \end{aligned}$$

we see that directional differentiability is immediate by continuity of $\tilde f$ following from that of $\tilde f_1$ by Proposition 31 above. Continuity of $\boldsymbol{\delta}_{\mathrm{Se}}\,\tilde f$ follows similarly.

Finally, for fixed F, G letting $\mathrm{P}\,(k)$ mean that the assertion of the Lemma to be established holds for k and for all the appropriate $E\,, f_1\,,$ we prove $\forall\, k \in I\!N_{\mathrm{o}}\,;\ \mathrm{P}\,(k)$ by induction as follows. The case $k = \emptyset$ or $k = 1\textbf{.}$ already having been established,

we assume $\mathrm{P}(k)$, and let g and g_1 be as above. Using Proposition 46 and assuming that $\tilde{f} \in \mathcal{D}^{k+1.}_{\text{Seip}}$, we have $\tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}}$ and $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$. From $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$ we get $(E \boxtimes E \boxtimes F, G, g_1{}^\wedge) \in \mathcal{D}^{k}_{\text{Seip}}$, whence by $\mathrm{P}(k)$ it follows that $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f}_1 = (\sigma_{\text{rd}}\, \boldsymbol{\delta}_{\text{Se}}\, \tilde{f}_1, g_1) \in \mathcal{D}^{k}_{\text{Seip}}$. Since by $\tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}}$ and $\mathrm{P}(1.)$ we have $\tilde{f}_1 \in \mathcal{D}^{1.}_{\text{Seip}}$, then $\tilde{f}_1 \in \mathcal{D}^{k+1.}_{\text{Seip}}$ follows. Conversely, letting $\tilde{f}_1 \in \mathcal{D}^{k+1.}_{\text{Seip}}$, we have $\tilde{f}_1 \in \mathcal{D}^{1.}_{\text{Seip}}$ and (x) that $\tilde{f}_1, \boldsymbol{\delta}_{\text{Se}}\, \tilde{f}_1 \in \mathcal{D}^{k}_{\text{Seip}}$. From (x) by $\mathrm{P}(k)$ it follows that $\tilde{f}, (E \boxtimes E \boxtimes F, G, g_1{}^\wedge) \in \mathcal{D}^{k}_{\text{Seip}}$. Since with $H = E \boxtimes F \boxtimes (E \boxtimes F)$ and $\ell_0 = \langle (x, v_1) : W = (x, v\,; u, v_1) \in \upsilon_s H \rangle$ and $\ell_1 = \langle (x, u, v) : W = (x, v\,; u, v_1) \in \upsilon_s H \rangle$ we have the continuous linear maps $(H, E \boxtimes F, \ell_0)$ and $(H, E \boxtimes E \boxtimes F, \ell_1)$ with the above established $g = \sigma_{\text{rd}}^2\, G \circ [\, g_1{}^\wedge \circ \ell_1, f \circ \ell_0\,]$, by (2), (4) and (5) of Proposition 47 it follows that $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f} = (\sigma_{\text{rd}}\, \boldsymbol{\delta}_{\text{Se}}\, \tilde{f}, g) \in \mathcal{D}^{k}_{\text{Seip}}$. Since by $\tilde{f}_1 \in \mathcal{D}^{1.}_{\text{Seip}}$ and $\mathrm{P}(1.)$ we have $\tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}}$, then $\tilde{f} \in \mathcal{D}^{k+1.}_{\text{Seip}}$ follows. ☐

54 Corollary. *Let* $\tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}}$. *Then* $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}} \Leftrightarrow D_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$ *for all* $k \in \mathbb{N}_{\text{o}}$.

Proof. Applying Lemma 53 above with E in place of F and F in place of G and $D_{\text{Se}}\, \tilde{f}$ in place of $\tilde{f}_1$ and $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f}$ in place of $\tilde{f}$, we see that under the assumption that $\tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}}$ we have $\boldsymbol{\delta}_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$ if and only if $D_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$. ☐

55 Theorem. *Let* $\tilde{f} = (E, F, f)$. *Then* $\tilde{f} \in \mathcal{D}^{k}_{\text{Seip}} \Leftrightarrow$ (a) *or* (b) *or* (c) *or* (d) *for*

(a) $k = \emptyset$ *and* $E, F \in \operatorname{scgVS}(\boldsymbol{R})$ *and*
$f \in F^{/E}$ *and* $\operatorname{dom} f \in \tau_{\text{rd}}\, E$ *and* f *continuous* $\tau_{\text{rd}}\, E \to \tau_{\text{rd}}\, F$,

(b) $k = 1.$ *and* $f : E \supset \operatorname{dom} f \to F$ *is* stetig differenzierbar
in the sense of [16; Definition 4.2, p. 60],

(c) $k \in \mathbb{N} \setminus 2.$ *and* $f : E \supset \operatorname{dom} f \to F$ *is* $k-$mal stetig differenzierbar
in the sense of [16; Definition 5.1, p. 75],

(d) $k = \infty$ *and* $f : E \supset \operatorname{dom} f \to F$ *is* unendlig oft stetig differenzierbar
in the sense of [16; Definition 6.1, p. 85].

Proof. First note that $\mathcal{D}^{k}_{\text{Seip}} = \emptyset$ if $k \notin \infty^+$. Then we see that the assertion concerning (a) is trivial, and that for (b) it is given in Corollary 45 (a) above. For a moment supposing we already also know (c) we obtain (d) by $\mathcal{D}^{1.}_{\text{Seip}} \subseteq \mathcal{D}^{0.}_{\text{Seip}}$ and $\mathcal{D}^{\infty}_{\text{Seip}} = \{\, \tilde{f} : \forall k \in \mathbb{N}_{\text{o}}\,;\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}} \}$ obtained in Proposition 46 above.

Now, for (c) considering the classes

$$\mathcal{D}_k = \{\, \tilde{f} : \exists\, \boldsymbol{f}\,;\, (\emptyset, \tilde{f}\,) \in \boldsymbol{f} \in \mathbf{U}^{\,k+1.} \text{ and } \forall\, i \in k\,;\, \boldsymbol{f}\,\grave{}\, i^+ = D_{\text{Se}}\, (\boldsymbol{f}\,\grave{}\, i) \},$$

since by the above we have $\mathcal{D}^{1}_{\text{Seip}\,72} \overset{*}{=} \{\, \tilde{f} : \tilde{f} \text{ is Seip}-\text{differentiable} \}$, in view of our definition of the Seip derivative, it suffices to prove that $\mathcal{D}^{k}_{\text{Seip}} = \mathcal{D}_k$ for all $k \in \mathbb{N}$. By $\overset{*}{=}$ already having the case $k = 1.$, assuming that with $k \in \mathbb{N}$ we have $\mathcal{D}^{k}_{\text{Seip}} = \mathcal{D}_k$, we prove that also $\mathcal{D}^{k+1.}_{\text{Seip}} = \mathcal{D}_{k+1.}$ as follows. By Proposition 46 (a) and Corollary 54 above, for all $\tilde{f}$ we have

$$\tilde{f} \in \mathcal{D}^{k+1.}_{\text{Seip}} \Leftrightarrow \tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}} \text{ and } \boldsymbol{\delta}_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}} \Leftrightarrow \tilde{f} \in \mathcal{D}^{1.}_{\text{Seip}} \text{ and } D_{\text{Se}}\, \tilde{f} \in \mathcal{D}^{k}_{\text{Seip}}$$
$$\Leftrightarrow \tilde{f} \in \mathcal{D}_{1.} \text{ and } D_{\text{Se}}\, \tilde{f} \in \mathcal{D}_k,$$

and as in the proof of Proposition 10 it is seen that $\tilde{f} \in \mathcal{D}_{1.}$ and $D_{\text{Se}}\, \tilde{f} \in \mathcal{D}_k \Leftrightarrow \tilde{f} \in \mathcal{D}_{k+1.}$. ☐

As a consequence of Seip's non-uniform manner of putting his definitions, the proof of our Theorem 55 above became a bit clumsy. See in particular [16; Definition 5.1, p. 75] whose clarity surely is not the best possible. We interpreted the vague presentation there so that in Theorem 55 we have the equivalence

$$\text{(c)} \Leftrightarrow k \in \mathbb{N} \setminus 2. \text{ and } (E, F, f) \in \mathcal{D}_k.$$

56 *Example.* We prove that $\tilde{f} \in \mathcal{D}^{\infty}_{\text{Seip}}$ for the map $\tilde{f} = (E \boxtimes E\,, \kappa_{\text{tv}}\grave{}\,E\,, f\,)$, where with $E = \mathcal{D}(I\!R)$ and $F = C^{\infty}(I\!R)$ and $\iota = \text{id}\; I\!R$, we have $f = f_1\,|\,(\upsilon_s E\,)^{\times 2\cdot}$ for the function $f_1 = \langle\, x \circ (\iota + y) : z = (x\,, y) \in (\upsilon_s F\,)^{\times 2\cdot}\,\rangle$.

We prove first that $(F \sqcap F, F, f_1) \in \mathcal{D}^{\infty}_{\text{Seip}}$. Note that since F is a Fréchet space, it is Seip−convenient and $F \sqcap F = F \boxtimes F$ holds. Letting G^H denote Seip's canonical function space $\mathcal{D}^{\infty}_{\text{Seip}}(\upsilon_s H_H\,, G)$, which in [16; Kapitel 6] would be written $\mathrm{D}^{\infty}(H \supset \upsilon_s H, G)$, we hence wish to have $f_1 \in \upsilon_s\big((\boldsymbol{R}^{\boldsymbol{R}})^{F \sqcap F}\big)$. By [16; Satz 6.18, pp. 91−92] this holds iff $^{\wedge}f_1 \in \upsilon_s\big(\boldsymbol{R}^{\boldsymbol{R} \sqcap (F \sqcap F)}\big)$. For $t \in I\!R$ and $x\,, y \in \upsilon_s F$ we have $^{\wedge}f_1\grave{}\,(t\,; x, y) = f_1\grave{}\,(x, y)\grave{}\,t = x\grave{}\,(t + (y\grave{}\,t))$, whence we see that $^{\wedge}f_1$ has the decomposition

$$(t\,; x, y) \mapsto (t, y, x) \mapsto (t, y\grave{}\,t, x) = (t, s, x) \mapsto (t + s, x) = (r, x) \mapsto x\grave{}\,r\,,$$

and consequently that for $^{\wedge}f_1 \in \upsilon_s\big(\boldsymbol{R}^{\boldsymbol{R} \sqcap (F \sqcap F)}\big)$, letting $\epsilon = \text{ev}\,|\,(I\!R \times (\upsilon_s F))$, it suffices that $\epsilon \in \upsilon_s\big(\boldsymbol{R}^{\boldsymbol{R} \sqcap F}\big)$. This in turn follows if we establish $^{\vee}\epsilon \in \upsilon_s\big((\boldsymbol{R}^{\boldsymbol{R}})^F\big)$ $= \upsilon_s\big(F^F\big)$. For $t \in I\!R$ and $x \in \upsilon_s F$ having $^{\vee}\epsilon\grave{}\,x\grave{}\,t = \epsilon\grave{}\,(t, x) = x\grave{}\,t$, we see that $^{\vee}\epsilon = \text{id}_{\text{v}} F$, hence that $^{\vee}\epsilon \in \upsilon_s\big(F^F\big)$ is trivial, in view of Proposition 47(2).

To obtain $\tilde{f} \in \mathcal{D}^{\infty}_{\text{Seip}}$, we deduce as follows. Letting $\mathrm{E}\,l = \mathcal{D}_{[-l,\,l\,]}(I\!R)$ and $\mathrm{G}\,l = \mathrm{E}\,l \sqcap \mathrm{E}\,l$ and $\tilde{\mathrm{f}}\,l = (\mathrm{G}\,l, \mathrm{E}\,l, f_1\,|\,\upsilon_s \mathrm{G}\,l)$, note that for every $l \in Z\!\!\!Z^+$ we have $\mathrm{E}\,l$ a closed topological linear subspace both in E and in F, and that for $x\,, y \in \upsilon_s \mathrm{E}\,l$ and $t \in I\!R \setminus [-l, l\,]$ we have $f_1\grave{}\,(x, y)\grave{}\,t = x\grave{}\,(t + (y\grave{}\,t)) = x\grave{}\,t = 0$, hence $f_1\grave{}\,(x, y) \in \upsilon_s \mathrm{E}\,l$, and further $\text{rng}\;\tau_{\text{rd}}\,\tilde{\mathrm{f}}\,l \subseteq \upsilon_s \mathrm{E}\,l$. From $(F \sqcap F, F, f_1) \in \mathcal{D}^{\infty}_{\text{Seip}}$ by Proposition 48 it follows by induction on $k \in I\!N_{\text{o}}$ that $\boldsymbol{\delta}^{\,k}_{\text{Se}}\,\tilde{\mathrm{f}}\,l$ is global with $\boldsymbol{\delta}^{\,k}_{\text{Se}}\,\tilde{\mathrm{f}}\,l \in \mathcal{C}_{\text{Se}\,0}$ for all $l \in Z\!\!\!Z^+$. Using this and noting that every $\tau_{\text{rd}}\,E-$compact K is $\tau_{\text{rd}}\,\mathrm{E}\,l-$compact for some $l \in Z\!\!\!Z^+$, it follows that $\boldsymbol{\delta}^{\,k}_{\text{Se}}\,\tilde{f}$ is global with $\boldsymbol{\delta}^{\,k}_{\text{Se}}\,\tilde{f} \in \mathcal{C}_{\text{Se}\,0}$ for $k \in I\!N_{\text{o}}$, whence by Proposition 48 we get $\tilde{f} \in \mathcal{D}^{\infty}_{\text{Seip}}$.

Seip systematically used the concept of being *scharf differenzierbar* in [16] when stating the premises in his various inverse and implicit function theorems. The next example shows that this property is so strong that these theorems are practically useless, at least when considering problems where maps of the form $x \mapsto \varphi \circ x$ between spaces of smooth functions are involved. Indeed, the only maps below of this form which are everywhere scharf differenzierbar are those given by some affine $\varphi : t \mapsto \alpha\, t + \beta$ with fixed $\alpha\,, \beta \in I\!R$.

57 *Example.* We fix a smooth nonaffine bijection $\varphi : I\!R \to I\!R$ with $0 \notin \text{rng}\,\varphi'$, and with $I = [\,0\,, 1\,]$ and $G = C^{\infty}(I)$ and $f = \langle\, \varphi \circ x : x \in \upsilon_s G\,\rangle$, consider the map $\tilde{f} = (G, G, f)$. Since G is Fréchet, it is Seip−convenient and $C^{\infty}_{\Pi}(\boldsymbol{R})\,|\,\{(G, G)\} \subseteq \mathcal{D}^{\infty}_{\text{Seip}}$, whence by [11; Remarks 3.7(a)] it follows that $\tilde{f}, (G, G, f^{-\iota}) \in \mathcal{D}^{\infty}_{\text{Seip}}$.

Note that we cannot obtain Seip−smoothness of $\tilde{f}$ and its inverse by (directly) applying any exponential law in Seip's theory since the closed interval I is not an admissible domain there. However, one can prove that there is a continuous linear map, extension operator $\epsilon : G \to E = C^{\infty}(I\!R)$ with $x \subseteq \epsilon\grave{}\,x$ for $x \in \upsilon_s G$. With the aid of Seip's exponential law, we obtain smoothness $E \to E$ of the map $f_1 : x \mapsto \varphi \circ x$. Since also $\rho : E \to G$ given by $y \mapsto y\,|\,I$ is a continuous linear map, in view of $f = \rho \circ f_1 \circ \epsilon$ we can get $\tilde{f} \in \mathcal{D}^{\infty}_{\text{Seip}}$, and similarly for the inverse.

We next show indirectly that for any fixed $\xi \in I\!R$ such that $\varphi''(\xi) \neq 0$, when we take $x = I \times \{\xi\}$, the above map $\tilde{f}$ is not scharf differenzierbar at the point x. Indeed, suppose that $\tilde{f}$ were scharf differenzierbar at x. This means that for

$$\varrho = \langle\, \varphi \circ (x + u) - \varphi \circ x - \varphi' \circ x \cdot u : u \in \upsilon_s G\,\rangle$$

it holds (s) that for every $\ell \in \mathcal{L}(G,G)$ there are open zero neighborhoods U,V in G such that for $h = \langle\, u + \ell\,\grave{}\,v : u \in U$ and $(u,v) \in \varrho\,\rangle$ we have a homeomorphism $h : \tau_{\mathrm{rd}}\, G \cap U \to \tau_{\mathrm{rd}}\, G \cap V$, and in particular a bijection $h : U \to V$. Note that for all $u \in \upsilon_s G$ we have $\varrho\,\grave{}\,u = \varphi \circ (x+u) - \varphi \circ x - \varphi' \circ x \cdot u =$

$$\int_0^1 (\varphi' \circ (x + s\,u) - \varphi' \circ x) \cdot u \,\mathrm{d}\,s = \int_0^1 \int_0^1 s\,\varphi'' \circ (x + s_1\, s\, u) \cdot u^2 \,\mathrm{d}\,s\,\mathrm{d}\,s_1\,.$$

We now consider $\ell = \langle\, u' : u \in \upsilon_s G\,\rangle$ in condition (s) and proceed to show that h is not even injective. Fixing $\varepsilon \in \mathbb{R}^+$ sufficiently small, for $u_0 = I \times \{\varepsilon\}$ we have $u_0 \in U$, and by $\varphi''(\xi) \neq 0$ we may also arrange that

$$(*) \qquad \int_0^1 \int_0^1 (s_1\, s^2 \varphi'''(\xi + s_1\, s\,\varepsilon)\,\varepsilon + 2\,s\,\varphi''(\xi + s_1\, s\,\varepsilon))\,\mathrm{d}\,s\,\mathrm{d}\,s_1 \neq 0\,.$$

We have $u_0 + (\varrho\,\grave{}\,u_0)' = u_0$, and defining the smooth $\chi : I \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned}(t\,;\eta\,,\eta') \mapsto \eta - \varepsilon + \int_0^1 \int_0^1 (s_1\, s^2 \varphi'''(\xi + s_1\, s\,\eta)\,\eta'\,\eta^2 \\ + 2\,s\,\varphi''(\xi + s_1\, s\,\eta)\,\eta\,\eta')\,\mathrm{d}\,s\,\mathrm{d}\,s_1\,,\end{aligned}$$

it now holds that $u_0 \in V$, and u_0 satisfies the differential equation $\chi \circ [\,\mathrm{id}\,; u\,, u'\,] = I \times \{0\}$ with initial condition $u\,\grave{}\,0 = \varepsilon$. More generally, we have

$$h^{-\iota}\,\grave{}\grave{}\,\{u_0\} = U \cap \{\, u : \chi \circ [\,\mathrm{id}\,; u\,, u'\,] = I \times \{0\}\}\,.$$

Consequently, we are done once we show that there is at least one $u \in U \setminus \{u_0\}$ satisfying the differential equation $\chi \circ [\,\mathrm{id}\,; u\,, u'\,] = I \times \{0\}$.

To establish this, we utilize our previous result [11; Theorem 5.2] from which it follows that if $0 \notin \{\, \partial_3 \chi\,\grave{}\,(t\,;\varepsilon\,,0) : t \in I\,\}$, there is γ with $(\boldsymbol{R}\,, G, \gamma) \in C_\Pi^\infty(\boldsymbol{R})$ and $(\varepsilon\,, u_0) \in \gamma$, and such that we have $\chi \circ [\,\mathrm{id}\,; u\,, u'\,] = I \times \{0\}$ and $u\,\grave{}\,0 = \eta$ whenever $(\eta, u) \in \gamma$. Then γ is in particular continuous $\tau_{\mathbb{R}} \to \tau_{\mathrm{rd}}\, G$, whence for all η sufficiently close to ε we have $\gamma\,\grave{}\,\eta = u \in U$, and here $u \neq u_0$ if we take $\eta \neq \varepsilon$ since $u_0\grave{}\,0 = \varepsilon \neq \eta = u\,\grave{}\,0$. By $(*)$ we indeed have $0 \notin \{\, \partial_3 \chi\,\grave{}\,(t\,;\varepsilon\,,0) : t \in I\,\}$.

Note also that essentially by the same method as above, we could have proved more generally that if $x \in \upsilon_s G$ only is such that $0 \notin \varphi''[\,\mathrm{rng}\, x\,]$, then $\tilde{f}$ is not scharf differenzierbar at x. In this case the function $p_3 = \langle\, \partial_3 \chi\,\grave{}\,(t\,;\varepsilon\,,0) : t \in I\,\rangle$ is not necessarily any more constant but we may still arrange $0 \notin \mathrm{rng}\, p_3$ by choosing $\varepsilon > 0$ sufficiently small. For example, if we take $\varphi = \langle\, t + \mathrm{e}^t : t \in \mathbb{R}\,\rangle$, then $\tilde{f}$ is not scharf differenzierbar at any x.

In the preceding example, we did not make explicit the space in which the function space integrals are taken in the expression for $\varrho\,\grave{}\,u$. Most conveniently, they are understood to be in $\boldsymbol{R}^{I\,]_{\mathrm{tvs}}}$, but they could have been considered also in G or $\boldsymbol{R}^{I\,]_{\mathrm{tvs}}}/_{\upsilon_s G}$, of which the latter is not sequentially complete.

58 *Example.* Let $E = (X, \mathcal{T}\,) \in \mathrm{LCS}(\boldsymbol{R})$, and let $\mathcal{C}$ be the set of bounded absolutely convex closed sets in E. If one wants to properly present the exact content of [16; Definition 8.6, p. 108], tracing the matters back to [16; Definitions 0.10, 8.3, pp. 4−5, 107−108] one arrives at the conclusion that a *full Seip norm* in E is (cf. Remark 3 above) for example some $\boldsymbol{\nu} = \bar{n}\,\boldsymbol{,}\, \bar{\varphi}_1\,\boldsymbol{,}\, \bar{\varphi}_2$ where $\bar{n}$ is a functor between certain small "additive" categories (it not being worth while specifying them here) and the $\bar{\varphi}_\iota$ are certain natural transformations between certain associated functors. However, it turns out that the $\bar{\varphi}_\iota$ once existing are unique, and that $\bar{n}$ is uniquely determined by its object component n for which we have (n) that $n \in \mathcal{C}^{\,\upsilon_s E}$ with $z \in n\,\grave{}\,z \subseteq \mathrm{Cl}_{\mathcal{T}}(n\,\grave{}\,x + n\,\grave{}\,(z-x))$ for all $x\,, z \in \upsilon_s E$. Conversely, every such n determines a unique full Seip norm $\boldsymbol{\nu}$. Hence, if one wants to keep matters as simple as possible, it is advisable to abandon the redundant [16; Definition 8.6] and instead define a *Seip norm* in E to be any n satisfying (n) above.

To give a nontrivial example of a Seip norm, taking $E = \boldsymbol{R}^{I\!R\,]\mathrm{tvs}}$ and $n = \langle\, I\!R^{I\!R} \cap \{\, z : \forall\, s \in I\!R\,;\, |\, z\grave{}s\,| \leq |\, x\grave{}s\,|\,\} : x \in \upsilon_s E\,\rangle$, then n is a Seip norm in E. Note that from the triangle inequality $|\, z\grave{}s\,| \leq |\, x\grave{}s\,| + |\, z\grave{}s - x\grave{}s\,|$ it easily follows that we even have $z \in n\grave{}z \subseteq n\grave{}x + n\grave{}(z - x)$ for $x\,, z \in \upsilon_s E$.

Supposing that also $\varphi \in I\!R^{I\!R \times I\!R}$ with $|\, \varphi\grave{}(s\,, t_1) - \varphi\grave{}(s\,, t_2)\,| \leq \frac{1}{2}\,|\, t_1 - t_2\,|$ for all $s\,, t_1\,, t_2 \in I\!R$, and considering $f = \langle\, \varphi \circ [\,\mathrm{id}\,, x\,] : x \in \upsilon_s E\,\rangle$, then we have $n\grave{}(f\grave{}x - f\grave{}y) \subseteq \frac{1}{2}\, n\grave{}(x - y)$ for all $x\,, y \in \upsilon_s E$. Consequently, by [16; Satz 8.8, p. 109] there is x with $\mathrm{dom}\,(f \cap \mathrm{id}) = \{x\}$. However, the same result can be obtained directly from Banach's fixed point theorem by considering for each fixed $s \in I\!R$ the $\{\,(s\,, t\,, |\, s - t\,|) : s\,, t \in I\!R\,\}$–contractor $\varphi\,(s\,, \cdot\,)$.

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